Lieb-Thirring and Cwickel-Lieb-Rozenblum inequalities for perturbed graphene with a Coulomb impurity

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Reference

Collaborator

This talk is based on joint results with Sergey Morozov (LMU Munich).

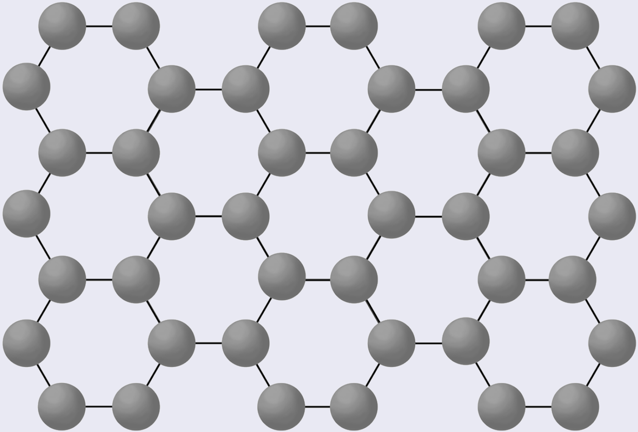
Preprints

The results presented in this talk can be found in two recent preprints on arXiv.

Graphene

Structure

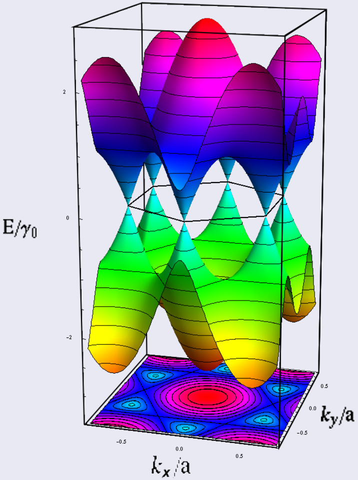
Graphene is a single atomic layer of graphite, in which carbon atoms are arranged in a honeycomb lattice.



Graphene

Zero gap semiconductor

The dispersion surfaces of the fully occupied valence and totally empty conduction bands touch at conical (Dirac) points. (Wallace 1947, Fefferman, Weinstein 2012)



The Coulomb-Dirac operator

Energy dispersion relation near the conical points

$$-\mathrm{i}\hbar v_F\, \boldsymbol{\sigma} \cdot \nabla \text{ with } \boldsymbol{\sigma} = (\sigma_1, \sigma_2) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \right)$$

Here $v_F \approx 10^6 m/s$ is the Fermi velocity. We choose units with $\hbar v_F = 1$.

Impurity

Suppose now that the graphene sheet contains an attractive Coulomb impurity of strength ν. The effective Hamiltonian is then given by

$$-\mathrm{i}\boldsymbol{\sigma} \cdot \nabla - \nu|\mathbf{x}|^{-1}.$$

For $\nu \in [0, 1/2]$ there exists a distinguished self-adjoint realization D_ν of this differential expression.

The model

State space

Since the Fermi energy is zero, the space of physically available states is $P_\nu^+ \mathsf{L}^2(\mathbb{R}^2;\mathbb{C}^2)$, where P_ν^+ is the spectral projector of D_ν to $[0,\infty)$.

Perturbed Coulomb-Dirac operator in the Furry picture

Consider an external potential V given by a Hermitian matrix-valued function. If it is not strong enough to substantially modify the Dirac sea, the perturbed effective Hamiltonian takes the form

$$D_\nu(V) := P_\nu^+(D_\nu - V)P_\nu^+.$$

Bound states

The negative spectrum of $D_\nu(V)$ may only consist of eigenvalues, which can be interpreted as bound states of a quantum dot. Here we prove estimates on these eigenvalues.

Results

Theorem 1 - Cwikel-Lieb-Rozenblum inequalities

Let $\nu \in [0, 1/2)$. There exists $C_\nu^{\mathrm{CLR}} > 0$ such that

$$\operatorname{rank}\left(D_\nu(V)\right)_- \leqslant C_\nu^{\mathrm{CLR}} \int_{\mathbb{R}^2} \operatorname{tr}\left(V_+(\mathbf{x})\right)^2 \mathrm{d}\mathbf{x}.$$

Theorem 2 - Virtual level at zero

Let

$$\widetilde{V}(r) := \frac{1}{2\pi} \int_0^{2\pi} \begin{pmatrix} V_{11}(r,\varphi) & -\mathrm{i} V_{12}(r,\varphi)\mathrm{e}^{\mathrm{i}\varphi} \\ \mathrm{i} V_{21}(r,\varphi)\mathrm{e}^{-\mathrm{i}\varphi} & V_{11}(r,\varphi) \end{pmatrix} \mathrm{d}\varphi.$$

Suppose that

$$\|\widetilde{V}\|_{\mathbb{C}^{2\times 2}} \in \mathsf{L}^1\big(\mathbb{R}_+, (1+r^2)\mathrm{d}r\big) \text{ and } \int_0^\infty \left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \widetilde{V}(r) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{C}^2} \mathrm{d}r > 0.$$

Then the negative spectrum of $D_{1/2}(V)$ is non-empty.

Results

Theorem 3 - Lieb-Thirring inequalities

Let $\nu \in [0, 1/2]$ and $\gamma > 0$. There exists $C^{\mathrm{LT}}_{\nu,\gamma} > 0$ such that

$$\operatorname{tr}\left(D_\nu(V)\right)_-^\gamma \leqslant C^{\mathrm{LT}}_{\nu,\gamma} \int_{\mathbb{R}^2} \operatorname{tr}\left(V_+(\mathbf{x})\right)^{2+\gamma} \mathrm{d}\mathbf{x}.$$

Remark

For $\nu = 1/2$ the inequality in Theorem 3 is a Hardy-Lieb-Thirring inequality.

Proofs of Theorems 1 and 3

Theorem 4

- For every $\nu \in [0, 1/2)$ there exists $C_\nu > 0$ such that
$$|D_\nu| \geqslant C_\nu \sqrt{-\Delta} \otimes \mathbb{1}_2 \tag{1}$$
holds.
- For any $\lambda \in [0, 1)$ there exists $K_\lambda > 0$ such that
$$|D_{1/2}| \geqslant \left(K_\lambda \ell^{\lambda-1} (-\Delta)^{\lambda/2} - \ell^{-1} \right) \otimes \mathbb{1}_2 \tag{2}$$
holds for any $\ell > 0$.

Thank you
for your attention!