On Proper Dissipative Extensions

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A few definitions

Definition (Dissipative operator)

Let A be a densely defined operator on a Hilbert space $\mathcal{H}$. We say that A is *dissipative* if and only if

$$\mathrm{Im}\langle\psi, A\psi\rangle \geq 0 \quad \text{for all } \psi \in \mathcal{D}(A)\,.$$

A few definitions

Definition (Dual pairs of operators and proper extensions)

Let (A, B) be a pair of densely defined and closable operators. We say that they form a *dual pair* if

$$A \subset B^* \quad \text{resp.} \quad B \subset A^* \,.$$

An extension A' of A is called a *proper* extension of the dual pair (A, B) if $A \subset A' \subset B^*$.

Remarks: Note that every dissipative operator is closable.

Examples:

- Let S be symmetric and T be bounded. Then $(S + T, S + T^*)$ is a dual pair and for example $S' + T$ would be a proper extension, where S' is a symmetric extension of S.
- Let A be a densely defined and closed operator. Then (A, A^*) is a dual pair and there is no non-trivial proper extension.

The problem for this talk

Given a dual pair of operators (A, B), where A and $(-B)$ are dissipative, how can we determine whether a proper extension A' of (A, B) is dissipative?

Motivation: singular differential operators like

- $i\frac{d}{dx} + i\frac{\gamma}{x}$ on $L^2(0,1)$ with $\gamma > 0$.
- $-\frac{d^2}{dx^2} + i\frac{\gamma}{x^2}$ on $L^2(0,1)$, where $\gamma > 0$.

The common core property

Definition (The common core property)

Let (A, B) be a dual pair of closed operators. We say that it has the common core property property if there exists a subspace $\mathcal{D} \subset \mathcal{D}(A) \cap \mathcal{D}(B)$ that is a core for both operators, i.e. if

$$A = \overline{A \restriction_{\mathcal{D}}} \quad \text{and} \quad B = \overline{B \restriction_{\mathcal{D}}}\,.$$

Remark:

- If (A, B) has the common core property, then the closures of $\mathcal{N}(A)$ and $\mathcal{N}(B)$ are complex conjugates.

The common core property

Examples:

- Let S be closed and symmetric. The dual pair (S, S) has the common core property.
- Let S be closed and symmetric and $V \geq 0$ be bounded. The dual pair $(S + iV, S - iV)$ has the common core property.
- Let $L_- f(x) := -if''(x) - \gamma \frac{f(x)}{x^2}$ and $L_+ f(x) := if''(x) - \gamma \frac{f(x)}{x^2}$. The dual pair of operators $(\overline{A_-}, \overline{A_+})$, where

$$A_\mp : \quad \mathcal{D}(A_\mp) = C_c^\infty(0,1) \\ f \mapsto L_\mp f$$

has the common core property by construction.

Main result

Some notation and assumptions:

- Let A be dissipative and (A, B) have the common core property and let $\mathcal{D}$ denote a common core.
- Let $\mathcal{V}$ be a subspace of $\mathcal{D}(B^*)$ such that $\mathcal{D}(A) \cap \mathcal{V} = \{0\}$. With $A_{\mathcal{V}}$ we mean the operator

$$A_{\mathcal{V}} : \quad \mathcal{D}(A_{\mathcal{V}}) = \mathcal{D}(A) \dot{+} \mathcal{V}$$
$$A_{\mathcal{V}} = B^* \restriction_{\mathcal{D}(A_{\mathcal{V}})} .$$

- The "imaginary part" V is defined to be the closure of $\frac{1}{2i}(A - B) \restriction_{\mathcal{D}}$.
- V_K denotes the self-adjoint Kreĭn-von Neumann extension of V. (Recall that for $V \geq \varepsilon > 0$, we have $\mathcal{D}(V_K) = \mathcal{D}(V) \dot{+} \ker V^*$.)

Theorem

$A_{\mathcal{V}}$ is dissipative if and only if $\mathcal{V} \subset \mathcal{D}(V_K^{1/2})$ and

$$\mathrm{Im}\langle v, B^* v\rangle \geq \|V_K^{1/2} v\|^2 \quad \text{for all } v \in \mathcal{V} .$$

A first order example

Let $\mathcal{H} = L^2(0,1)$, $0 < \gamma < 1/2$ and consider the dual pair of operators:

$$A_{0,\pm}: \quad \mathcal{D}(A_{0,\pm}) = \mathcal{C}_c^\infty(0,1)$$
$$(A_{0,\pm}f)(x) = if'(x) \pm \frac{i\gamma}{x} f(x)\,.$$

With $A := \overline{A_{0,+}}$ and $B := \overline{A_{0,-}}$, the dual pair (A, B) has the common core property. It can be shown that

$$\mathcal{D}(B^*) = \mathcal{D}(A) \dot{+} \operatorname{span}\{x^{-\gamma}, x^{\gamma+1}\}\,.$$

The imaginary part $\frac{1}{2i}(A - B)$ is the essentially self-adjoint multiplication operator by γx^{-1} on $\mathcal{C}_c^\infty(0,1)$. Thus, $V_K^{1/2}$ is just the maximal multiplication operator by $\sqrt{\gamma} x^{-1/2}$.

Since $x^{-\gamma} \notin \mathcal{D}(V_K^{1/2})$, the only possible candidate for a maximally dissipative extension of A is $x^{\gamma+1}$ and it can be checked that it is indeed.

A second order example

Let $\mathcal{H} = L^2(0,1)$, $\gamma \geq \sqrt{3}$ and consider the dual pair of operators:

$$A_{0,\pm} : \quad \mathcal{D}(A_{0,\pm}) = \mathcal{C}_c^\infty(0,1)$$
$$(A_{0,\pm} f)(x) = \pm i f''(x) - \frac{\gamma}{x^2} f(x) \,.$$

Define $A := \overline{A_{0,-}}$ and $B := \overline{A_{0,+}}$. A calculation shows that

$$\mathcal{D}(B^*) = \mathcal{D}(A) \dot{+} \text{span} \left\{ x^\omega, x^{\overline{\omega}+2} \right\} \,,$$

where $\omega = (1 + \sqrt{1 + 4i\gamma})/2$. The "imaginary part" is $V = -\frac{d^2}{dx^2}$ with domain $\mathcal{C}_c^\infty(0,1)$ and it can be shown that $\mathcal{D}(V_K^{1/2}) = H^1(0,1)$ and

$$\| V_K^{1/2} f \|^2 = \| f' \|^2 - |f(1) - f(0)|^2 \,.$$

A second order example

By an elementary linear transformation, we can construct functions $\psi(x)$ and $\phi(x)$ such that

$$\mathrm{span}\{x^{\omega}, x^{\overline{\omega}+2}\} = \mathrm{span}\{\psi, \phi\}$$

and $\psi(1) = \phi'(1) = 1$ as well as $\psi'(1) = \phi(1) = 0$. Applying the theorem, we find that the operators A_ρ given by

$$A_\rho : \qquad \mathcal{D}(A_\rho) = \mathcal{D}(A) \dot{+} \mathrm{span}\{\rho\psi + \phi\}, \quad A_\rho = B^* \restriction_{\mathcal{D}(A_\rho)}$$

are dissipative if and only if

$$|\rho - 1/2| \geq 1/2\,.$$

Reference

C. F., S. Naboko, and I. Wood: The Proper Dissipative Extensions of a Dual Pair, Integr. Equ. Oper. Theory, August 2016, Volume 85, Issue 4, pp 573–599.

Thanks for your attention!