The Complete Set of Infinite Volume Ground States for Kitaev's Abelian Quantum Double Models

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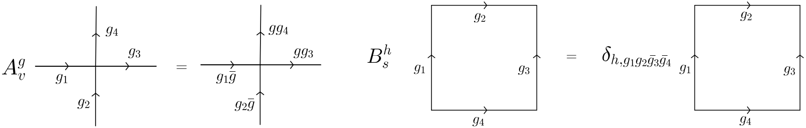
Kitaev's Quantum Double Model (Kitaev, 2003)

Let G be a finite group and $\mathcal{B}$ the bond set of $\mathbb{Z}^2$. For $e \in \mathcal{B}$, assign a $|G|$-dimensional Hilbert space with an orthonormal basis labeled g.

Interaction terms are defined for each star v and plaquette f by

$$A_v = \frac{1}{|G|} \sum_{g \in G} A_v^g, \qquad \text{and} \qquad B_f = B_f^e,$$

where



The local Hamiltonian for $\Lambda \subset \mathcal{B}$ a finite subset is:

$$H_\Lambda = \sum_{v \in \mathcal{V}_\Lambda} (I - A_v) + \sum_{f \in \mathcal{F}_\Lambda} (I - B_f)$$

When $G = \mathbb{Z}_2$ this is the well known toric code model. Some general properties of the q.d. models include:

- The interactions terms form a commuting family of projectors.
- The space of ground states is frustration-free and topologically ordered, when defined on a surface of genus g the ground state degeneracy only depends on g, local observables cannot distinguish between ground states (local topological quantum order).
- Excitations occur when one of the f.f. conditions are violated. Ribbon operators generate excitations at their endpoints from a ground state. The excitations are anyons, that is, they obey braided statistics. If G is abelian, they can be labeled by a pair $(\chi, c) \in \widehat{G} \times G$, where $\widehat{G}$ is the group of characters.

Thermodynamic Limit

The quasi-local algebra of observables is

$$\mathcal{A} = \overline{\bigcup_{\Lambda \subset_f \mathcal{B}} \mathcal{A}_\Lambda}^{\|\cdot\|} \quad \text{where} \quad \mathcal{A}_\Lambda = \bigotimes_{e \in \Lambda} M_2(\mathbb{C}).$$

The dynamics is given by a one-par. group of automorphisms of $\mathcal{A}$,

$$\tau_t = e^{it\delta} \quad \text{where} \quad \delta(\cdot) = \lim_{L \to \infty} [H_{\Lambda_L}, \cdot\,]$$

where the limit is in the strong sense and Λ_L is any monotone sequence absorbing $\mathcal{B}$, e.g., Λ_L is the bond set of $[-L, L]^2$. The local algebra of observables, $\mathcal{A}_{loc} = \bigcup_{\Lambda \subset_f \mathcal{B}} \mathcal{A}_\Lambda$ is a core for δ.

A state is a linear functional $\omega : \mathcal{A} \to \mathbb{C}$ such that $\omega(I) = 1$ and $\omega(A) \geq 0$ if $A \geq 0$.

Definition

A state ω is called an infinite volume ground state if

$$\omega(A^*\delta(A)) \geq 0 \quad \textit{for all} \quad A \in \mathcal{A}_{loc}.$$

Let K denote the set of all infinite volume ground states.

- This definition expresses that local perturbations do not decrease the energy of a ground state.
- Infinite volume ground states are often obtained as weak* limit of finite volume ground states. The choice of finite volume boundary conditions play a crucial role.

A ground state satisfying the conditions $\omega(A_v) = 1$ and $\omega(B_f) = 1$ for all stars v and plaquettes f is called frustration-free.

Theorem (Alicki-Fannes-Horodecki (2007), Naaijkens (2011), Fiedler-Naaijkens (2015))

There exists a unique translation invariant ground state ω^0 of the quantum double model. ω^0 satisfies the following properties:

- *ω^0 is the unique frustration free ground state.*
- *ω^0 is a pure state.*
- *Let $(\pi_0, \Omega_0, \mathcal{H}_0)$ be a GNS-representation for ω^0 and H_0 be the Hamiltonian in this GNS representation. Then, spec$(H_0) = \mathbb{Z}^{\geq 0}$ with a simple ground state eigenvector Ω_0.*

In particular, ω^0 is a gapped ground state.

Superselection Sectors

Now let G be a finite abelian group.

Naaijkens (2011), Fiedler-Naaijkens (2015) constructed single excitation states in the thermodynamic limit. These states are labelled by their charge type and position of the charge, $\omega_s^{\chi,c}$.

Furthermore, distinct charge labels correspond to inequivalent states, and hence different superselection sectors. The superselection structure is completely described by the representation theory of the q.d., $\text{Rep}(\mathcal{D}(G))$.

The single excitation states $\omega_s^{\chi,c}$ are infinite volume ground states. Indeed, the set of ground states decomposes into $|G|^2$ sectors corresponding to the different charge types.

Theorem (C-Naaijkens-Nachtergaele, 2016)

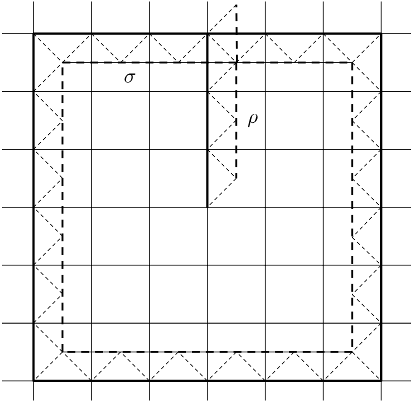
Let $\omega \in K$ be a ground state of the q.d. model for abelian group G. Then there exists disjoint subsets $K^{\chi,c}$ of K such that ω has the convex decomposition

$$\omega = \sum_{\chi \in \widehat{G}, c \in G} \lambda_{\chi,c}(\omega)\omega^{\chi,c} \qquad \text{where} \quad \omega^{\chi,c} \in K^{\chi,c}.$$

Furthermore,

- *For the single excitation states, $\omega_s^{\chi,c} \in K^{\chi,c}$*
- *$K^{\chi,c}$ is a face in the set of all states. In particular, if $\omega^{\chi,c}$ is an extremal point of $K^{\chi,c}$ then $\omega^{\chi,c}$ is a pure state.*
- *If $\omega^{\chi,c} \in K^{\chi,c}$ is a pure state then it is equivalent to the single excitation state $\omega_s^{\chi,c}$.*

The strategy of the proof is to reduce the infinite volume calculation to a finite volume calculation. In particular, we find a boundary term for every box such that the restriction of any infinite volume ground state to the box is a ground state of the finite volume Hamiltonian plus the boundary term.



Concluding Remarks

- The complete ground state problem has been solved for the XY-chain by Araki-Matsui (1985), for the XXZ-chain by Matsui (1996) and Koma-Nachtergaele (1998), and for the finite-range spin chains with a unique f.f. MPS ground state by Ogata (2016). Our result is the first solution to the ground state problem for a quantum model in two-dimensions.
- A current challenge in mathematical physics is the classification of gapped ground state phases. One approach is to construct a complete set of invariants. The invariance of the structure of anyon quasi-particles is usually taken as fact. Our results are a first step in rigorously studying the stability properties of the superselection structure of quantum double models.