Large Deviation Principles for Weakly Interacting Fermions

N. J. B. Aza

Departamento de Física Matemática, Universidade de São Paulo

Joint work with J.-B. Bru, W. de Siqueira Pedra and L. C. P. A. M. Müssnich

October 08, 2016

Large Deviation Theory and Quantum Lattice Systems

Lebowitz–Lenci–Spohn '00, Gallavotti–Lebowitz–Mastropietro '02, Netočny–Redig '04, Lenci–Rey-Bellet '05, Hiai–Mosonyi–Ogawa '07, Ogata '10, Ogata–Rey-Bellet '11, de Roeck–Maes–Netočny–Schütz '15

Large Deviation Theory and Quantum Lattice Systems

Lebowitz–Lenci–Spohn '00, Gallavotti–Lebowitz–Mastropietro '02, Netočny–Redig '04, Lenci–Rey-Bellet '05, Hiai–Mosonyi–Ogawa '07, Ogata '10, Ogata–Rey-Bellet '11, de Roeck–Maes–Netočny–Schütz '15

- Observe that for ρ a state on the C^*–algebra $\mathfrak{A}$ and $A \in \mathfrak{A}$ a selfadjoint element, there is a unique probability measure $\mu_{\rho,A}$ on $\mathbb{R}$ such that $\mu_{\rho,A}(\mathrm{spec}(A)) = 1$ and, for all continuous functions $f : \mathbb{R} \to \mathbb{C}$,

$$\rho(f(A)) = \int_{\mathbb{R}} f(x)\mu_{\rho,A}(\mathrm{d}x).$$

- $\mu_A \doteq \mu_{\rho,A}$ is the measure associated to ρ and A. For a sequence of selfadjoints $\{A_l\}_{l\in\mathbb{R}^+}$ of $\mathfrak{A}$, and a state ρ, we say that these satisfy a Large Deviation Principle (LDP), with scale $|\Lambda_l|$, if, for all Borel measurable $\Gamma \subset \mathbb{R}$,

$$-\inf_{x\in\mathring{\Gamma}} \mathscr{I}(x) \leq \liminf_{l\to\infty} \frac{1}{|\Lambda_l|} \log \mu_{A_l}(\Gamma) \leq \limsup_{l\to\infty} \frac{1}{|\Lambda_l|} \log \mu_{A_l}(\Gamma) \leq -\inf_{x\in\overline{\Gamma}} \mathscr{I}(x)$$

Large Deviation Theory and Quantum Lattice Systems

- To find an LDP we desire to use the Gärtner–Ellis Theorem (GET) to μ_{A_l}, through the scaled cumulant generating function

$$\overline{f}(s) = \lim_{l\to\infty} \frac{1}{|\Lambda_l|} \log \rho(\mathrm{e}^{s|\Lambda_l|A_l}), \quad s \in \mathbb{R}.$$

- If $\overline{f}$ exists and is differentiable, then the good rate function $\mathscr{I}$ is the Legendre–Fenchel transform of $\overline{f}$.
- In the case of lattice fermions we represent $\overline{f}$ as a Berezin–integral and analyse it using "tree expansions". The scale $|\Lambda_l|$ will be then the volume of the boxes Λ_l:

$$\Lambda_l \doteq \{(x_1, \ldots, x_d) \in \mathbb{Z}^d : |x_1|, \ldots, |x_d| \leq l\} \in \mathscr{P}_{\mathrm{f}}(\mathbb{Z}^d).$$

- For lattice fermions, $\mathfrak{A}$ is the CAR C^*–algebra generated by the identity $\mathbb{1}$ and $\{a_{\mathfrak{s},x}\}_{\mathfrak{s},x\in\mathfrak{L}}$. $\mathfrak{L} \doteq \mathrm{S} \times \mathbb{Z}^d$ where S is the set of Spins of single fermions. However, our proofs do not depend on the particular choice of S.

Large Deviation Theory and Quantum Lattice Systems

- CAR:

$$\{a_x, a_{x'}\} = 0, \quad \{a_x, a_{x'}^*\} = \delta_{x,x'}\mathbb{1}.$$

- $\mathfrak{A}_\Lambda \subset \mathfrak{A}$ is the C^*–subalgebra generated $\mathbb{1}$ and $\{a_x\}_{x\in\Lambda}$.
- An interaction Φ is a map $\mathscr{P}_f(\mathbb{Z}^d) \to \mathfrak{A}$ s.t. $\Phi_\Lambda = \Phi_\Lambda^* \in \mathfrak{A}^+ \cap \mathfrak{A}_\Lambda$ and $\Phi_\emptyset = 0$.
- Φ is of finite range if for $\Lambda \in \mathscr{P}_f(\mathbb{Z}^d)$ and some $R > 0$, $\mathrm{diam}\,\Lambda > R$ $\to \Phi_\Lambda = 0$.
- For any interaction Φ, we define the space average $K_l^\Phi \in \mathfrak{A}_{\Lambda_l}$ by

$$K_l^\Phi \doteq \frac{1}{|\Lambda_l|} \sum_{\Lambda\in\mathscr{P}_f(\mathbb{Z}^d),\, \Lambda\in\Lambda_l} \Phi_\Lambda.$$

Main Result

Note that finite range interactions define equilibrium (KMS) states of $\mathfrak{A}$.

Theorem (A., Bru, Müssnich, Pedra)

Let $\beta > 0$ and consider any finite range translation invariant interaction $\Psi = \Psi_0 + \Psi_1$. If the interparticle component Ψ_1 (Ψ_0 is the free part) is small enough (depending on β), then any invariant equilibrium state ρ of Ψ and the sequence of averages K_l^{Φ} of ANY translation invariant interaction Φ, have an LDP and $s \mapsto \overline{f}(s)$ is analytic at small s.

Main Result

Remarks

1. Note that, in contrast to previous results, we do not impose β to be small or Φ (defining K_l^{Φ}) to be an one–site interaction.
2. Uniqueness of KMS states is not used.
3. Use C^*–algebras formalism and Grassmann algebras.
4. Determinant bounds or study of Large Determinants.
5. Direct representation of $\overline{f}$ by Berezin–integrals. In particular we do not use the correlation functions.
6. Beyond the LDP, the analyticity of $\overline{f}(\cdot)$ together with the Bryc Theorem implies the Central Limit Theorem for the system.

Main Result

Sketch of the proof.

1

$$\overline{f}(s) = \lim_{l\to\infty} \lim_{l'\to\infty} \frac{1}{|\Lambda_l|} \log \frac{\mathrm{tr}(\mathrm{e}^{-\beta H_{l'}} \mathrm{e}^{sK_l})}{\mathrm{tr}(\mathrm{e}^{-\beta H_{l'}})}.$$

Main Result

Sketch of the proof.

1

$$\overline{f}(s) = \lim_{l\to\infty} \lim_{l'\to\infty} \frac{1}{|\Lambda_l|} \log \frac{\mathrm{tr}(e^{-\beta H_{l'}} e^{sK_l})}{\mathrm{tr}(e^{-\beta H_{l'}})}.$$

2 From a Feynmann–Kac–like formula for traces, we write the KMS state as a Berezin–integral

$$\frac{\mathrm{tr}_{\wedge^*\mathfrak{H}}(e^{-\beta H_{l'}} e^{sK_l})}{\mathrm{tr}_{\wedge^*\mathfrak{H}}(e^{-\beta H_{l'}^{(0)}})} = \lim_{n\to\infty} \int d\mu_{C_{l'}^{(n)}}(\mathfrak{H}^{(n)}) e^{\mathscr{W}_{l,l'}^{(n)}}.$$

Main Result

Sketch of the proof.

1
$$\bar{f}(s) = \lim_{l\to\infty} \lim_{l'\to\infty} \frac{1}{|\Lambda_l|} \log \frac{\mathrm{tr}(\mathrm{e}^{-\beta H_{l'}} \mathrm{e}^{sK_l})}{\mathrm{tr}(\mathrm{e}^{-\beta H_{l'}})}.$$

2 From a Feynmann–Kac–like formula for traces, we write the KMS state as a Berezin–integral
$$\frac{\mathrm{tr}_{\wedge^*\mathfrak{H}}(\mathrm{e}^{-\beta H_{l'}} \mathrm{e}^{sK_l})}{\mathrm{tr}_{\wedge^*\mathfrak{H}}(\mathrm{e}^{-\beta H_{l'}^{(0)}})} = \lim_{n\to\infty} \int \mathrm{d}\mu_{C_{l'}^{(n)}}(\mathfrak{H}^{(n)}) \mathrm{e}^{\mathscr{W}_{l,l'}^{(n)}}.$$

3 The covariance $C_{l'}^{(n)}$ satisfies:
$$\left| \det \left[(\varphi_a^*)^{(k_a)} \left(C_{l'}^{(n)} \left(\varphi_b^{(k_b)} \right) \right) \right]_{a,b=1}^{m} \right| \leq \left(\prod_{a=1}^{m} \|\varphi_a^*\|_{\mathfrak{H}^*} \right) \left(\prod_{b=1}^{m} \|\varphi_b\|_{\mathfrak{H}} \right).$$

Main Result

Sketch of the proof.

1
$$\overline{f}(s) = \lim_{l\to\infty} \lim_{l'\to\infty} \frac{1}{|\Lambda_l|} \log \frac{\mathrm{tr}(\mathrm{e}^{-\beta H_{l'}} \mathrm{e}^{sK_l})}{\mathrm{tr}(\mathrm{e}^{-\beta H_{l'}})}.$$

2 From a Feynmann–Kac–like formula for traces, we write the KMS state as a Berezin–integral

$$\frac{\mathrm{tr}_{\wedge^*\mathfrak{H}}(\mathrm{e}^{-\beta H_{l'}} \mathrm{e}^{sK_l})}{\mathrm{tr}_{\wedge^*\mathfrak{H}}(\mathrm{e}^{-\beta H_{l'}^{(0)}})} = \lim_{n\to\infty} \int \mathrm{d}\mu_{C_{l'}^{(n)}}(\mathfrak{H}^{(n)}) \mathrm{e}^{\mathscr{W}_{l,l'}^{(n)}}.$$

3 The covariance $C_{l'}^{(n)}$ satisfies:

$$\left|\det\left[(\varphi_a^*)^{(k_a)}\left(C_{l'}^{(n)}\left(\varphi_b^{(k_b)}\right)\right)\right]_{a,b=1}^{m}\right| \leq \left(\prod_{a=1}^{m} \|\varphi_a^*\|_{\mathfrak{H}^*}\right)\left(\prod_{b=1}^{m} \|\varphi_b\|_{\mathfrak{H}}\right).$$

Use Brydges–Kennedy Tree expansions (BKTE) to verify GET. BKTE are solution of an infinite hierarchy of coupled ODEs. . .

End

Perspectives and Questions

Perspectives:

1 Quantum Hypothesys Testing? Open problems, e.g., study thermodynamic limit of the relative entropy between equilibrium state $\omega_\Lambda^\beta \in \mathfrak{A}_\Lambda$ and translation invariant state ω_Λ.

2 Related problems to our approach.

3 ...

Perspectives and Questions

Perspectives:

1 Quantum Hypothesys Testing? Open problems, e.g., study thermodynamic limit of the relative entropy between equilibrium state $\omega_\Lambda^\beta \in \mathfrak{A}_\Lambda$ and translation invariant state ω_Λ.

2 Related problems to our approach.

3 ...

Open Questions:

1 LDP for time correlation (transport coefficients)?

2 Systems in presence of disorder?

3 What about LDP for commutators of averages $\mathrm{i}[K^{\Phi_1}, K^{\Phi_2}]$ in place of simple averages K^Φ? (Also related to transport)

4 ...

Thank you!

Supporting facts

1. For any invertible operator $C \in \mathscr{B}(\mathfrak{H})$ and $\xi \in \wedge^*(\mathfrak{H} \oplus \bar{\mathfrak{H}})$, the Gaussian Grassmann integral: $\int \mathrm{d}\mu_C(\mathfrak{H}) : \wedge^* \left(\mathfrak{H} \oplus \bar{\mathfrak{H}}\right) \to \mathbb{C}\mathbf{1}$ with covariance C, is defined by

$$\int \mathrm{d}\mu_C(\mathfrak{H})\xi \doteq \det(C) \int \mathrm{d}\left(\mathfrak{H}\right) \mathrm{e}^{\langle \mathfrak{H}, C^{-1}\mathfrak{H}\rangle} \wedge \xi.$$

2. $\int \mathrm{d}\mu_C(\mathfrak{H})\mathbf{1} = \mathbf{1}$ and for any $m, n \in \mathbb{N}$ and all $\bar{\varphi}_1, \ldots, \bar{\varphi}_m \in \bar{\mathfrak{H}}$, $\varphi_1, \ldots, \varphi_n \in \mathfrak{H}$,

$$\int \mathrm{d}\mu_C(\mathfrak{H})\bar{\varphi}_1 \cdots \bar{\varphi}_m \varphi_1 \cdots \varphi_m = \det\left[\bar{\varphi}_k(C\varphi_l)\right]_{k,l=1}^m \delta_{m,n}\mathbf{1}$$

3. For all $N \in \mathbb{N}$ and $A_0, \ldots, A_{N-1} \in \mathscr{B}(\wedge^*\mathfrak{H})$,

$$\mathrm{Tr}_{\wedge^*\mathfrak{H}}(A_0 \cdots A_{N-1})\mathbf{1} = \left(\prod_{k=0}^{N-1} \int \mathrm{d}\left(\mathfrak{H}^{(k)}\right)\right) \mathrm{E}_{\mathfrak{H}}^{(N)} \left(\prod_{k=0}^{N-1} \varkappa^{(k)}(A_k)\right),$$

where $\mathrm{E}_{\mathfrak{H}}^{(N)} \doteq \mathrm{e}^{\langle \mathfrak{H}^{(0)}, \mathfrak{H}^{(0)}\rangle + \langle \mathfrak{H}^{(0)}, \mathfrak{H}^{(N-1)}\rangle + \sum_{k=1}^{N-1}(\langle \mathfrak{H}^{(k)}, \mathfrak{H}^{(k)}\rangle - \langle \mathfrak{H}^{(k)}, \mathfrak{H}^{(k-1)}\rangle)}$,
$\varkappa^{(k)} \doteq \varkappa_{(0,0)}^{(k,k)} \circ \varkappa : \mathscr{B}(\wedge^*\mathfrak{H}) \to \wedge^*(\mathfrak{H}^{(k)} \oplus \bar{\mathfrak{H}}^{(k)})$ and for $i, j, k, l \in \{0, \ldots, N\}$, $\varkappa_{(i,j)}^{(k,l)} : \wedge^*(\mathfrak{H}^{(i)} \oplus \bar{\mathfrak{H}}^{(j)}) \to \wedge^*(\mathfrak{H}^{(k)} \oplus \bar{\mathfrak{H}}^{(l)})$.