SPECTRAL STATISTICS OF RANDOM SCHRÖDINGER OPERATORS WITH NON-ERGODIC RANDOM POTENTIAL

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The Model

- On $\ell^2(\mathbb{Z}^d)$ define the operator Δ by

$$(\Delta u)(n) = \sum_{|k|_+=1} u(n+k) - 2d\, u(n), \quad u \in \ell^2(\mathbb{Z}^d).$$

- The random potential V^ω is the multiplication operator on $\ell^2(\mathbb{Z}^d)$ given by

$$(V^\omega u)(n) = (1 + |n|^\alpha)\, \omega_n\, u(n), \quad \alpha > 0$$

  where $(\omega_n)_{n\in\mathbb{Z}^d}$ are iid real random variables uniformly distributed on $[0, 1]$.

- Consider the probability space $\left(\Omega = [0,1]^{\mathbb{Z}^d}, \mathcal{B}_\Omega, \mathbb{P} = \otimes\mu\right)$. The random operators H^ω on $\ell^2(\mathbb{Z}^d)$

$$H^\omega = -\Delta + V^\omega, \quad \omega = (\omega_n)_{n\in\mathbb{Z}^d} \in \Omega.$$

Gordon-Jakšić-Molchanov-Simon

- Define $\{a_j\}_{j\geq 1}$, $(a_0 = \infty)$ given by

$$a_j = \inf_{\substack{\|\phi\|=1 \\ \phi \in C(A_j)}} \sum_{\substack{(n,m) \\ |n-m|_+=1}} |\phi(n) - \phi(m)|^2, \;\; \phi \in \ell^2(\mathbb{Z}^d),$$

  $A_j \subset \mathbb{Z}^d$ with $\# A_j = j$ and A_j are connected.

- A path between points $n, m \in \mathbb{Z}^d$ is a sequence of sites

$$\tau = (n_1, n_2, \cdots, n_k), \;\; n_1 = n, \;\; n_k = m, |n_{j+1} - n_j|_+ = 1.$$

  $X \subset \mathbb{Z}^d$ is connected if any two points in X can be connected with a path which lies within X.

- $\{a_j\}_{j\geq 1}$ is a strictly decreasing sequence .

$$a_j < a_{j-1}, \; j = 1, 2, \cdots \cdots$$

The Spectrum (Gordon-Jakšić-Molchanov-Simon)

- H^ω has discrete spectrum a.e ω if and only if $\alpha > d$.
- If $N^\omega(E)$ denotes the number of eigenvalues of H^ω which are less than E then for $\alpha > d$ and for a.e ω

$$N^\omega(E) = O(E^{\frac{d}{\alpha}}) \;\; as \;\; E \to \infty.$$

Fixed a $k \in \mathbb{N}$ and $d/k \geq \alpha > d/(k+1)$ for a.e ω

- $\sigma(H^\omega) = \sigma_{pp}(H^\omega)$
- $\sigma_{ess}(H^\omega) = [a_k, \infty)$,
- $\#\sigma_{disc}(H^\omega) < \infty$.

IDS (Gordon-Jakšić-Molchanov-Simon)

- Define $H_L^\omega = \chi_L H^\omega \chi_L$, χ_L is the projection onto $\ell^2(\Lambda_L)$, Λ_L is a cube with side length $(2L+1)$ centered at origin.
- $N_L^\omega(E) = \#\{j : \ \ E_j \leq E, \ \ E_j \in \sigma(H_L^\omega)\}$
- If $d/k > \alpha > d/(k+1), \ k \in \mathbb{N}$ and $E \in (a_j, a_{j-1}), \ 1 \leq j \leq k$, then

$$\lim_{L\to\infty} \frac{N_L^\omega(E)}{L^{d-j\alpha}} = N_j(E) \ \ (Non\ random)\ a.e\, \omega,$$

- If $\alpha = d/k$ and $E \in (a_j, a_{j-1})$, $1 \leq j < k$, the above is valid. If $E \in (a_k, a_{k-1})$ then

$$\lim_{L\to\infty} \frac{N_L^\omega(E)}{lnL} = N_k(E) \ \ (Non\ random) \ \ a.e\, \omega.$$

- For each ω, H_L^ω is a matrix (symmetric) of order $(2L+1)^d$.
- $\#\sigma(H_L^\omega) = (2L+1)^d$.
- In other words average spacing between two consecutive eigenvalue of H_L^ω inside (a_j, a_{j-1}) is of order $L^{-(d-j\alpha)}$.
- Now we want study how the eigenvalues of H_L^ω are accumulating (are they following any rule) insdie (a_j, a_{j-1}), as L gets large.

Problem Formulation (Dolai-Mallick)

- Define $\xi^{\omega}_{L,E}(\cdot) = \sum_j \delta_{L^{d-\alpha}(E_j - E)}(\cdot), \quad E \in (a_1, \infty)$.
- $\xi^{\omega}_{L,E}(I) = \#\{j : E_j \in E + L^{-(d-\alpha)} I\} \quad I \subset \mathbb{R}$
- $\{\xi^{\omega}_{L,E}\}$ is a sequence of integer valued random measure (i.e sequence of Point processs).
- We want study the Weak limit of $\xi^{\omega}_{L,E}$

Our result (Dolai-Mallick)

- For $d \geq 3$, $\max\{2, \frac{d}{2}\} < \alpha < d$ and $E \in S$ the sequence of point process $\{\xi^{\omega}_{L,E}\}_L$ converges weakly to the Poisson point processs with intensity measure $N_1'(E)\ dx$.
- $\lim_{L\to\infty} \mathbb{P}\big(\omega : \xi^{\omega}_{L,E}(B) = n\big) = e^{-N_1'(E)|B|} \frac{\big(N_1'(E)|B|\big)^n}{n!}, \ n \in \mathbb{N} \cup \{0\}, \ \ B \textit{ is bounded Borel set of } \mathbb{R}.$
- In above $S \subset (a_1, \infty)$ such that $N_1'(E) > 0, \ \ E \in S$

Idea of the proof (Exponential decay of Green's function)

- We divide Λ_L into N_L^d numbers of disjoint cubes C_p with side length $\frac{2L+1}{N_L}$

$$\Lambda_L = \cup C_p, \quad |C_p| = \left(\frac{2L+1}{N_L}\right)^d, \ N_L = O(2L+1)^{\epsilon}, \ 0 < \epsilon < 1.$$

- Let H_p^{ω} be the restriction of H^{ω} to C_p. Define

$$\eta_{p,L,E}^{\omega}(\cdot) = \sum_{x \in \sigma(H_p^{\omega})} \delta_{L^{d-\alpha}(x-E)}$$

- Using Aizenman-Molcanov method we showed that

$$\mathbb{E}\left(\left|G_{\Lambda}^{\omega}(n,m;z)\right|^s\right) \leq Ce^{-r|n-m|}, \quad z \in \mathbb{C}^+$$

$$G_{\Lambda}^{\omega}(n,m;z) = \langle \delta_n, (H_{\Lambda}^{\omega} - z)^{-1}\delta_m \rangle, \ \Lambda \subset \mathbb{Z}^d$$

- $\mathbb{E}^{\omega}\left| \int f d\xi^{\omega}_{L,E} - \sum_{p=1}^{N_L^d} \int f d\eta^{\omega}_{p,L,E} \right| \xrightarrow{L\to\infty} 0, \quad f \in C_c^+(\mathbb{R})$
- Since collection of functions of the form $\frac{1}{x-z}, \quad z \in \mathbb{C}^+$ is dense in C_c^+ so, to show the above convergence it is enough to verify the following

$$\left| \int \frac{1}{x-z} d\xi^{\omega}_{L,E}(x) - \sum_{p=1}^{N_L^d} \int \frac{1}{x-z} d\eta^{\omega}_{p,L,E}(x) \right| \to 0 \ \ as \ \ L \to \infty.$$

- Now the above will follows from exponential decay of Green's functions.

Convergence of uniformly asymptotically negligible triangural array

Now we will show $\sum_{p=1}^{N_L^d} \eta^{\omega}_{p,E,L}$ converges weakly to the Poisson point process with intensity measure $N_1' dx$.
To show this it is enough to verify the following three conditions, for any bounded interval I

- (Con 1)
$$\sup_{1 \leq p \leq N_L^d} \mathbb{P}\left(\omega : \eta^{\omega}_{p,L,E}(I) > \epsilon\right) = 0 \quad as \quad L \to \infty \quad \forall \ \epsilon > 0.$$
- (Con 2) $\sum_{p=1}^{N_L^d} \mathbb{P}\left(\omega : \eta^{\omega}_{p,L,E}(I) \geq 2\right) = 0 \ \ as \ \ L \to \infty$
- (Con 3) $\sum_{p=1}^{N_L^d} \mathbb{P}\left(\omega : \eta^{\omega}_{p,L,E}(I) \geq 1\right) = N_1'(E)|I| \ \ as \ \ L \to \infty.$

Minami and Wegner Estimate (Combes-Germinet-Klein)

- Wegner Estimate

$$\mathbb{E}\big(TrE_{H_L^\omega}(I)\big) \leq \sum_{n\in\Lambda_L} \big(1+|n|^\alpha\big)^{-1} |I|$$

- Minami Estimate

$$\mathbb{E}\Big(TrE_{H_L^\omega}(I)\big(TrE_{H_L^\omega}(I)-1\big)\Big) \leq \Big(\sum_{n\in\Lambda_L} \big(1+|n|^\alpha\big)^{-1} |I|\Big)^2$$

- $\nu_L(\cdot) = \frac{1}{L^{d-\alpha}}\mathbb{E}^\omega\big(Tr\big(E_{H_L^\omega}(\cdot)\big)\big), \quad N_1(x) = \nu(a_1, x), \quad x > a_1$

$$\nu_L \xrightarrow[weakly]{L\to\infty} \nu \quad on \quad (a_1, \infty).$$

- From Wegner estimate it will follow

$$\nu_L(I) \leq C|I| \;\; and \;\; \nu(I) \leq |I|.$$

Existence of Intensity and it's positivity

- Convergence of densities inside (a_1, ∞)

$$f_L(E) = \frac{d\nu_L}{dx}(E) \xrightarrow[L\to\infty]{uniformly} N_1'(E) = \frac{d\nu}{dx}(E) \ \ on \ [E-\delta, E+\delta], \ \ \delta > 0.$$

- To show the above convergence we first showed that $\psi_L(z) = \frac{1}{L^{d-\alpha}} \sum_{n\in\Lambda_L} \mathbb{E}^\omega\big(G^\omega_{\Lambda_L}(n, n; z)\big)$ is analytic and uniformly bounded on a region $G(\subset \mathbb{C})$ which contain $S \subset (a_1, \infty)$.
- The density f_L is given by $f_L(E) = \frac{d\nu_L}{dx}(E) = \frac{1}{\pi} Im\psi_L(E + i0)$.

- We showed that
  $\nu(a,b) = N_1(b) - N_1(a) > 0, \;\; |b-a| > 4d,$
  $a, \; b \in (4d, \infty) \subset (a_1, \infty)$
- The above can be shown using the min-max principal and the operator inequality $A^{\omega}_{L,0} \leq H^{\omega}_{L} \leq A^{\omega}_{L,4d}$.
- $A^{\omega}_{L,0} = \sum_{n \in \Lambda_L} b_n \omega_n, \;\; A^{\omega}_{L,4d} = 4d + \sum_{n \in \Lambda_L} b_n \omega_n, \, b_n = 1 + |n|^{\alpha}.$

- (Con 1) will follow from Wegner estimate.

$$\begin{aligned}\mathbb{P}\left(\eta^{\omega}_{L,p,E}(I) \geq 1\right) &\leq \mathbb{E}^{\omega}\left[\eta^{\omega}_{L,p,E}(I)\right] \\ &= \mathbb{E}^{\omega}\left[TrE_{H^{\omega}_{C_p}}\left(E + L^{-(d-\alpha)}I\right)\right] \\ &= O(N_L^{-(d-\alpha)})\end{aligned}$$

- (Con 2) will follow from Minami estimate.

$$\sum_{j\geq 2} \mathbb{P}\left(\eta^{\omega}_{L,p,E}(I) \geq j\right) \leq \mathbb{E}^{\omega}\left[\eta^{\omega}_{L,p,E}(I)\left(\eta^{\omega}_{L,p,E}(I) - 1\right)\right]$$

$$= \mathbb{E}^{\omega}\Bigg[TrE_{H^{\omega}_{C_p}}\left(E + L^{-(d-\alpha)}I\right) \times$$

$$\left\{ TrE_{H^{\omega}_{C_p}}\left(E + L^{-(d-\alpha)}I\right) - 1 \right\}\Bigg]$$

$$= O\left(N_L^{-2(d-\alpha)}\right).$$

- (Con 3) will follow from the following identity and convergence of density
- $\mathbb{P}\left(\eta^{\omega}_{L,p,E}(I) \geq 1\right) = \mathbb{E}^{\omega}\left[\eta^{\omega}_{L,p,E}(I)\right] - \sum_{j\geq 2}\mathbb{P}\left(\eta^{\omega}_{L,p,E}(I) \geq j\right)$

$$\begin{aligned}
\lim_{L\to\infty}\sum_{p}\mathbb{E}^{\omega}\left[\eta^{\omega}_{L,p,E}(I)\right] &= \lim_{L\to\infty}\mathbb{E}^{\omega}\left[\xi^{\omega}_{L,E}(I)\right] \\
&= \lim_{L\to\infty}\mathbb{E}^{\omega}\left(TrE_{H^{\omega}_{\Lambda L}}\left(E + L^{-(d-\alpha)}I\right)\right) \\
&= \lim_{L\to\infty} L^{d-\alpha}\nu_L\left(E + L^{-(d-\alpha)}\right) \\
&= \lim_{L\to\infty} L^{d-\alpha}\int_{E+L^{-(d-\alpha)}I} f_L(x)dx \\
&= \lim_{L\to\infty}\int_I f_L\left(E + L^{-(d-\alpha)}y\right)dy = N_1'(E)|I|.
\end{aligned}$$

Reference

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Thank You