Transport and Entanglement in Disordered XY Chains

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QMath13, Georgia Tech

October 2016

Overview

- The XY chain.
- Dynamical entanglement.
- Particle number transport.
- Energy transport.

$$H = -\sum_{j=1}^{n-1} \mu_j [(1+\gamma_j)\sigma_j^x \sigma_{j+1}^x + (1-\gamma_j)\sigma_j^y \sigma_{j+1}^y] - \sum_{j=1}^{n} \nu_j \sigma_j^z$$

- $\Lambda = [1, n]$, Λ_0 a block of spins (subinterval of Λ).
- The Hilbert space: $\mathcal{H} := \bigotimes_{x \in \Lambda} \mathcal{H}_x = (\mathbb{C}^2)^{\otimes n}, \quad \dim \mathcal{H} = 2^n$.
- μ_j, γ_j and ν_j are i.i.d.

$\downarrow$ **Jordan-Wigner** $\downarrow$

$$H = \mathcal{C}^* M \mathcal{C}, \ \mathcal{C} := (c_1, c_1^*, c_2, c_2^*, \dots, c_n, c_n^*)^t.$$

M is the block Jacobi matrix

$$M := \begin{pmatrix} -\nu_1 \sigma^z & \mu_1 S(\gamma_1) & & \\ \mu_1 S(\gamma_1)^t & \ddots & \ddots & \\ & \ddots & \ddots & \mu_{n-1} S(\gamma_{n-1}) \\ & & \mu_{n-1} S(\gamma_{n-1})^t & -\nu_n \sigma^z \end{pmatrix},$$

$$S(\gamma) = \begin{pmatrix} 1 & \gamma \\ -\gamma & -1 \end{pmatrix}, \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Assumptions:

- The XY chain H has almost sure simple spectrum.
- M satisfies eigencorrelator localization, i.e $\mathbb{E}\left(\sup_{|g|\leq 1} \|g(M)_{jk}\|\right) \leq C_0(1+|j-k|)^{-\beta}$, for some $\beta > 6$.

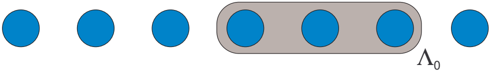
Applications:

$\mu_j = \mu$, $\gamma_j = \gamma$ for all $j \in \mathbb{N}$.
ν_j are i.i.d from an absolutely continuous, compactly supported distribution.

- Isotropic case ($\gamma = 0$): $M \longrightarrow$ Anderson Model.
- Anisotropic case ($\gamma \neq 0$):
  - Large disorder case. **Elgart/Shamis/Sodin (2012).**
  - Uniform spectral gap for M around zero. **Chapman /Stolz (2014).**

Dynamical Entanglement

The Entanglement Entropy and the Entanglement of Formation



Fix $\Lambda_0 \subseteq \Lambda$, consider the decomposition:

$$\mathcal{H} = \mathcal{H}_{\Lambda_0} \otimes \mathcal{H}_{\Lambda\setminus\Lambda_0}, \text{ where } \mathcal{H}_{\Lambda_0} = \bigotimes_{x\in\Lambda_0} \mathcal{H}_x, \quad \mathcal{H}_{\Lambda\setminus\Lambda_0} = \bigotimes_{x\in\Lambda\setminus\Lambda_0} \mathcal{H}_x. \tag{1}$$

Let ρ be a pure state in $\mathcal{B}(\mathcal{H})$, then

$$\mathcal{E}(\rho) = -\operatorname{Tr}\left[\rho^1 \log \rho^1\right], \quad \text{where} \quad \rho^1 = \operatorname{Tr}_{\mathcal{H}_2} \rho.$$

For any (mixed) state $\rho \in \mathcal{B}(\mathcal{H})$, then

$$E_f(\rho) = \inf_{p_k,\psi_k} \sum_k p_k \mathcal{E}\left(|\psi_k\rangle\rangle\psi_k|\right).$$

Dynamical Entanglement

Motivation Question

- For $1 \leq \ell \leq n$, let $H_{[1,\ell]}$ and $H_{[\ell+1,n]}$ be the restrictions of H to the corresponding interval.
- Let $\rho^{(1)}$ and $\rho^{(2)}$ be any eigenstates/thermal states of $H_{[1,\ell]}$ and $H_{[\ell+1,n]}$, respectively.
- We study $\rho_t := e^{-itH}\left(\rho^{(1)} \otimes \rho^{(2)}\right)e^{itH}$.
- ρ_t is an entangled state with respect to $\mathcal{H}_{[1,\ell]} \otimes \mathcal{H}_{[\ell+1,n]}$.

Question:

What can we say about the entanglement of ρ_t?

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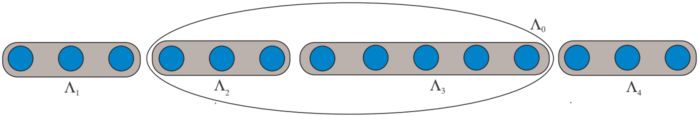
Dynamical Entanglement

Problem Setting

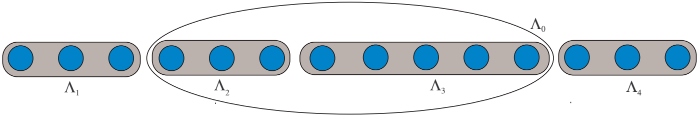


In general

- Decompose Λ into disjoint intervals $\Lambda_1, \Lambda_2, \ldots, \Lambda_m$.
- H_{Λ_k} is the restriction of H to Λ_k.
- ψ_k is an eigenfunction of H_{Λ_k}, and $\rho_k = |\psi_k\rangle\langle\psi_k|$.
- Define $\rho = \bigotimes_{k=1}^m \rho_k$, and its dynamics $\rho_t = e^{-itH}\rho e^{itH}$.

Dynamical Entanglement: Main Theorem

Dynamics of products of eigenstates

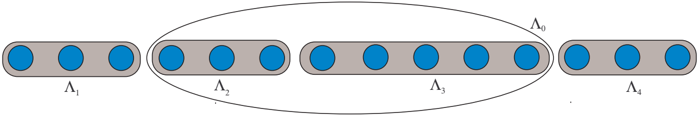


Theorem

There exists $C < \infty$ *such that*

$$\mathbb{E}\left(\sup_{t,\{\psi_k\}_{k=1,2,\dots,m}} \mathcal{E}(\rho_t) \right) \leq C$$

for all n, m, *any choice of the interval* $\Lambda_0 \subset \Lambda$ *and all decompositions* $\Lambda_1, \dots, \Lambda_m$ *of* $\Lambda = [1, n]$.



- ρ_{β_k} is a thermal state of H_{Λ_k}.
- Define $\rho_\beta = \bigotimes_{k=1}^m \rho_{\beta_k}$, and its dynamics $(\rho_\beta)_t = e^{-itH} \rho_\beta e^{e^{itH}}$.

Result:

$$\mathbb{E}\left(\sup_{t,\beta} E_f\left((\rho_\beta)_t\right)\right) \leq C$$

If $m = n$

- number of decompositions is n.
- eigenfunctions are up and down spins: $e_\uparrow := |\uparrow\rangle$ and $e_\downarrow := |\downarrow\rangle$.

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{\uparrow, \downarrow\}^n$, the up-down configuration associated with α is given by:

$$e_\alpha = e_{\alpha_1} \otimes e_{\alpha_2} \otimes \ldots \otimes e_{\alpha_n}$$

Result: $\boxed{\mathbb{E}\left(\sup_\alpha \mathcal{E}(e^{-itH}|e_\alpha\rangle\langle e_\alpha|e^{itH})\right) < C.}$ **Barderson, Pollman, and Moore** (2012).

For $m = 1$ (No Decomposition)

Let ψ be an eigenfunction of the full XY chain H.

Result: $$\mathbb{E}\left(\sup_{\psi} \mathcal{E}(|\psi\rangle\langle\psi|)\right) < C.$$ **Pastur/Slavin** (2014). **AR/Stolz** (2015).

Let ρ_β be a thermal state of the full XY chain H.

Result: $$\mathbb{E}\left(\sup_{\beta} E_f(\rho_\beta)\right) < C.$$

Particle Number Transport

An Isotropic XY Chain in Random Transversal Magnetic Field

$$H_{\mathsf{iso}} = -\sum_{j=1}^{n-1}[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y] - \sum_{j=1}^{n}\nu_j\sigma_j^z$$

$\downarrow$ **Jordan-Wigner** $\downarrow$

$H_{\mathsf{iso}} = c^*Ac + \left(\sum_j \nu_j\right)\mathbb{1}$, where $c := (c_1, c_2, \dots, c_n)^t$.

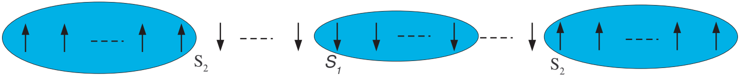
$$A := \begin{pmatrix} -\nu_1 & \mu & & \\ \mu & \ddots & \ddots & \\ & \ddots & \ddots & \mu \\ & & \mu & -\nu_n \end{pmatrix}, \quad \mathbb{E}\left(\sup_{|g|\leq 1} |\langle e_j, g(A)e_k\rangle|\right) \leq Ce^{-\eta|j-k|}.$$

Particle Number Transport

The Particle Number Operator

$$\mathcal{N} := \sum_{j\in\Lambda} |e_\uparrow\rangle\langle e_\uparrow|_j \text{ and } \mathcal{N}_S := \sum_{j\in S} |e_\uparrow\rangle\langle e_\uparrow|_j.$$

- $\mathcal{N} e_\alpha = k e_\alpha$, where $k = |\{j \ : \alpha_j = \uparrow\}|$.
- Let $\rho = |e_\alpha\rangle\langle e_\alpha|$ then $\langle \mathcal{N} \rangle_\rho := \mathrm{Tr}\, \mathcal{N} \rho = k$ is the expected number of up-spins.
- $[H, \mathcal{N}] = 0 \Rightarrow$ The number of up-spins is conserved in time.
- $\rho_t = e^{-itH_{\text{iso}}} \rho e^{itH_{\text{iso}}}$ is the time evolution of ρ.
- $\langle \mathcal{N}_S \rangle_{\rho_t}$ is the expected number of up-spins in S at time t.



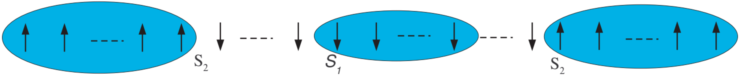
- Fix $S_1 \subset \Lambda$ and $S_2 \subset \Lambda \setminus [\min S_1, \max S_1]$.
- Initial state: $\rho = \bigotimes_{j=1}^{n} \begin{pmatrix} \eta_j & 0 \\ 0 & 1-\eta_j \end{pmatrix}$, with $\eta_j = 0$ for all $j \notin S_2$.

$$\mathbb{E}\left(\sup_t \langle \mathcal{N}_{S_1} \rangle_{\rho_t}\right) \leq \frac{4C}{(1+e^{-\eta})^2} e^{-\eta dist(S_1,S_2)}$$

Similar results for disordered Tonks-Girardeau gas, **Seiringer/Warzel** (2016).

Energy Transport

Isotropic Case



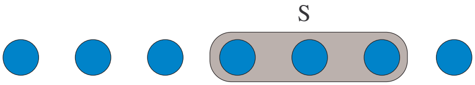
- Fix $S_1 = [a, b] \subset \Lambda$ and $S_2 \subset \Lambda \setminus S_1$.
- Initial state: $\rho = \bigotimes_{j=1}^{n} \begin{pmatrix} \eta_j & 0 \\ 0 & 1-\eta_j \end{pmatrix}$, with $\eta_j = 0$ for all $j \notin S_2$.

$$\boxed{\mathbb{E}\left(\sup_t \left|\langle H_{S_1}\rangle_{\rho_t} - \langle H_{S_1}\rangle_{\rho}\right|\right) \leq \frac{4CD}{(1+e^{-\eta})^2} e^{-\eta dist(S_1,S_2)}},$$

where $D = \sup_n \|A_n\|$.

Energy Transport

Anisotropic Case



- Fix $S = [a, b] \subset \Lambda$.
- H_S is the restriction of the XY chain to S.
- Initial state: $\rho = \bigotimes_{j=1}^{n} \begin{pmatrix} \eta_j & 0 \\ 0 & 1 - \eta_j \end{pmatrix}$.

$$\boxed{\mathbb{E}\left(\sup_t |\langle H_S \rangle_{\rho_t} - \langle H_S \rangle_\rho|\right) \leq \tilde{C}},$$

Thank you.