

Quantum analogues of geometric inequalities for Information Theory

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based on a joint work with Robert Koenig and Stefan Huber

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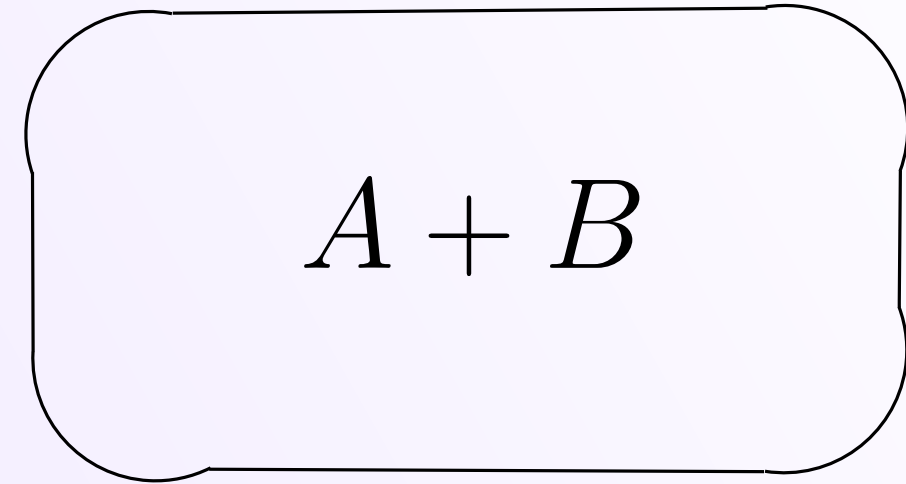
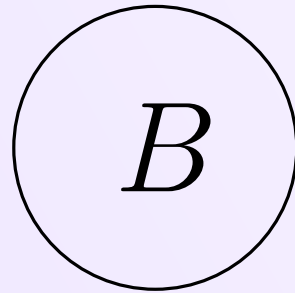
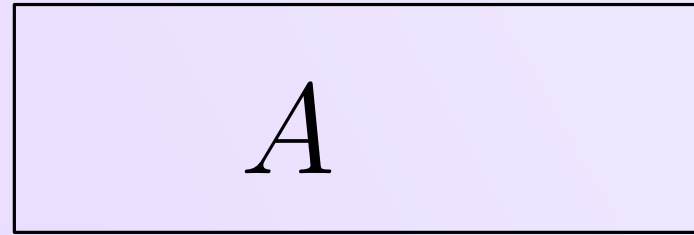
Outline of the talk

- Introduction
 - Geometric inequalities in information theory
 - Classical vs quantum inequalities
- Quantum inequalities
 - Entropy power inequality
 - Concavity of the entropy power of diffusion semigroup
 - Quantum isoperimetric inequality
- Application to the quantum Ornstein-Uhlenbeck semigroup
- Optimizing entropy rates of the quantum attenuator semigroup

Geometric inequalities

$$\mathbb{R}^n = \mathbb{R}^2$$

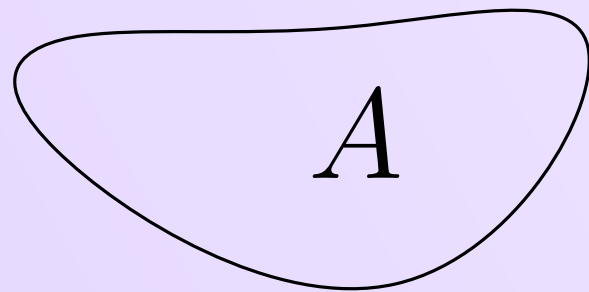
Brunn-Minkowski inequality



$$A + B = \{a + b \mid a \in A, b \in B\}$$

$$\text{vol}(A)^{1/n} + \text{vol}(B)^{1/n} \leq \text{vol}(A + B)^{1/n}$$

Isoperimetric inequality



$$\text{area}(A) \leq \frac{1}{4\pi} \text{length}(\partial A)^2$$

Geometry vs Information theory

Set $A \subset \Omega$

volume $\text{vol}(A)$

Random variable X on Ω
with prob. mass function $p_x = \Pr[X = x]$

entropy power $2^{H(X)}$

Shannon entropy $H(X) = -\sum p_x \log p_x$

Geometry vs Information theory

Set $A \subset \Omega$

volume $\text{vol}(A)$

Consider a set $A^n = A \times \cdots \times A \in \Omega^n$

$$\text{vol}(A^n) = (\text{vol}(A))^n$$

Random variable X on Ω

with prob. mass function $p_x = \Pr[X = x]$

entropy power $2^{H(X)}$

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Consider n i.i.d. with $P_{X^n} = P_X \cdots P_X$

has the following property:

$$\forall \epsilon > 0 \quad \exists M_{n,\epsilon} \subset \Omega^n \text{ s.t.}$$

$$|M_{n,\epsilon}| \sim (2^{H(X)})^n$$

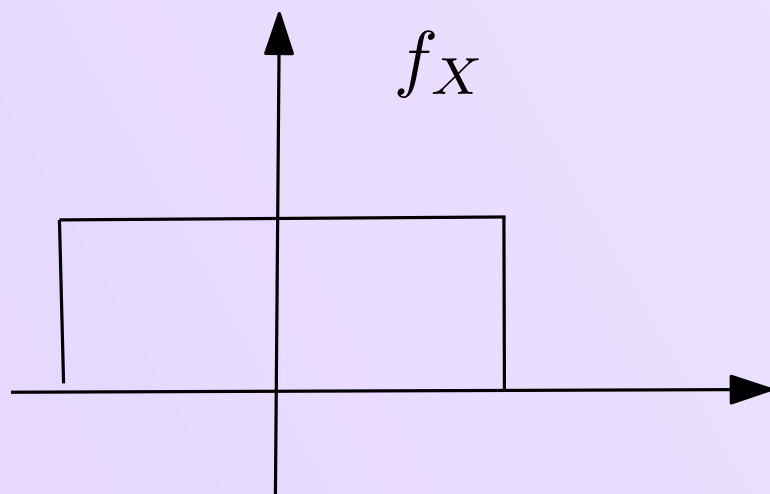
$$\text{and } \Pr[X^n \in M_{n,\epsilon}] \geq 1 - \epsilon$$

Geometry vs Information theory

Set $A \in \mathbb{R}^n$

volume $\text{vol}(A)$

addition $A + B$



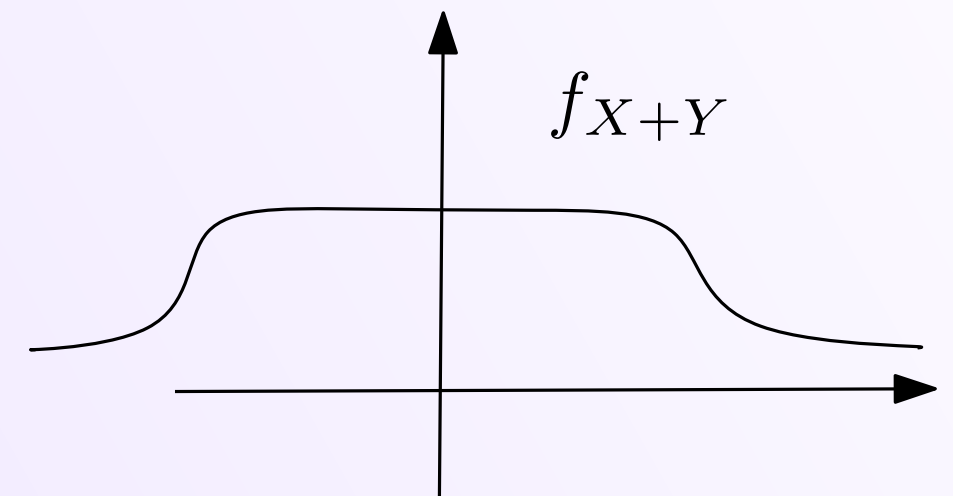
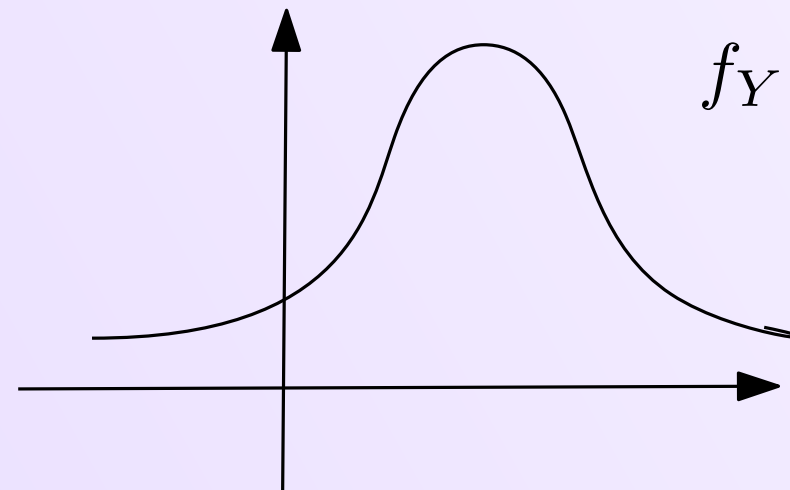
Random variable X on \mathbb{R}^n
with prob. density function f_X

entropy power $e^{2H(X)/n}$

entropy $H(X) = - \int_{\mathbb{R}^n} f_X(x) \log f_X(x) dx$

convolution: $X + Y$ has a density function

$$f_{X+Y}(x) = \int f_X(x-z)f_Y(z)dz$$



Geometric inequalities

Brunn-Minkowski inequality

$$\text{vol}(A)^{1/n} + \text{vol}(B)^{1/n} \leq \text{vol}(A + B)^{1/n}$$

Shannon's entropy power inequality ['48]

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Geometric inequalities

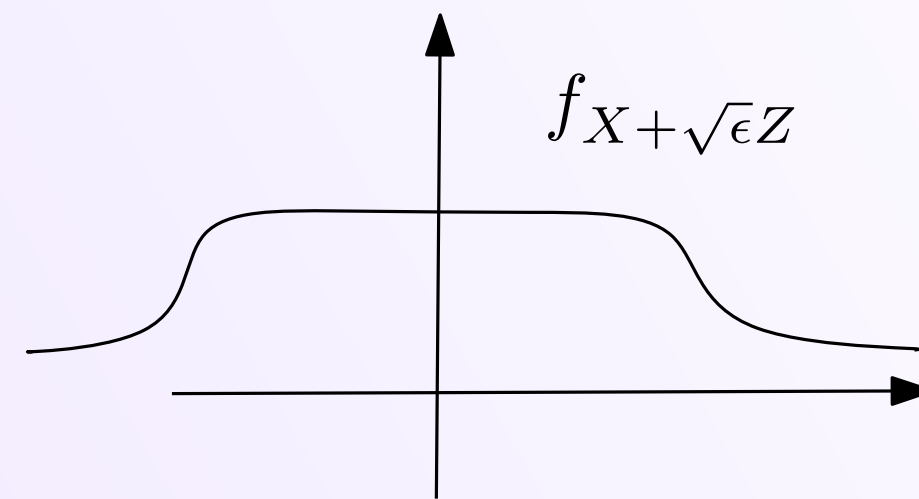
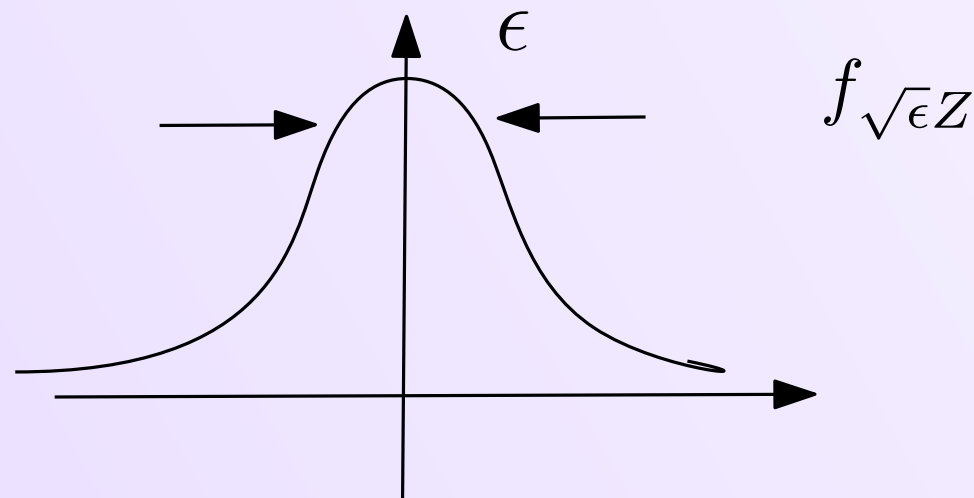
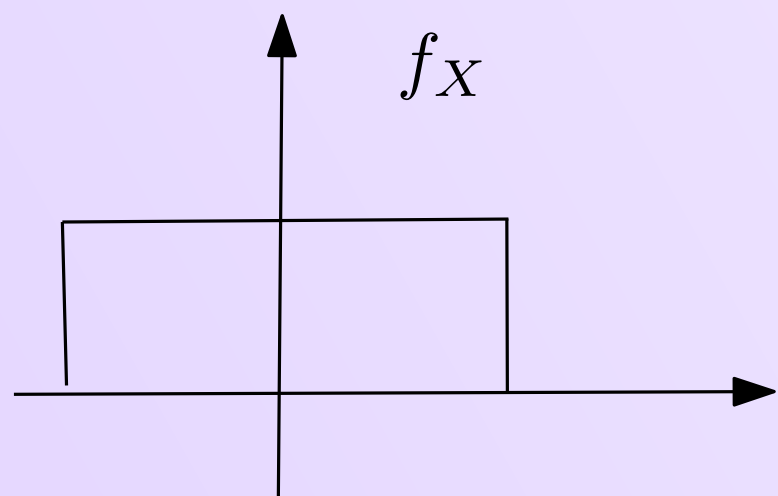
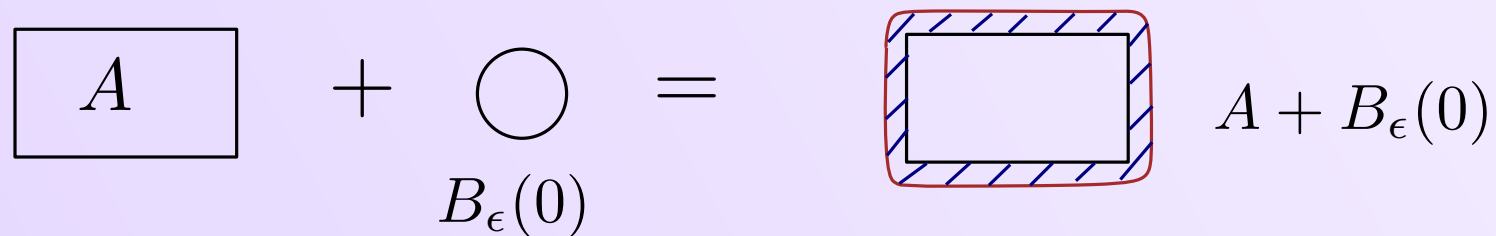
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Isoperimetric inequality

$$\text{area}(A) \leq \frac{1}{4\pi} \text{length}(\partial A)^2$$

$$\text{length}(\partial A) = \lim_{\epsilon \rightarrow 0} \frac{\text{area}(A + B_\epsilon(0)) - \text{area}(A)}{\epsilon}$$



Shannon's entropy power inequality ['48]

$$e^{2H(X)/n} + e^{2H(Y)/n} \leq e^{2H(X+Y)/n}$$

“length” $\lim_{\epsilon \rightarrow 0} \frac{e^{2H(X+\sqrt{\epsilon}Z)/n} - e^{2H(X)/n}}{\epsilon}$

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$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} e^{2H(X+\sqrt{\epsilon}Z)/n} = \frac{2}{n} e^{2H(X)/n} \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} H(X + \sqrt{\epsilon}Z)$$

Fisher information

$$J(X) = 2 \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} H(X + \sqrt{\epsilon}Z)$$

Geometric inequalities

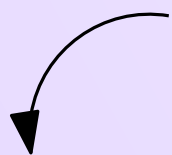
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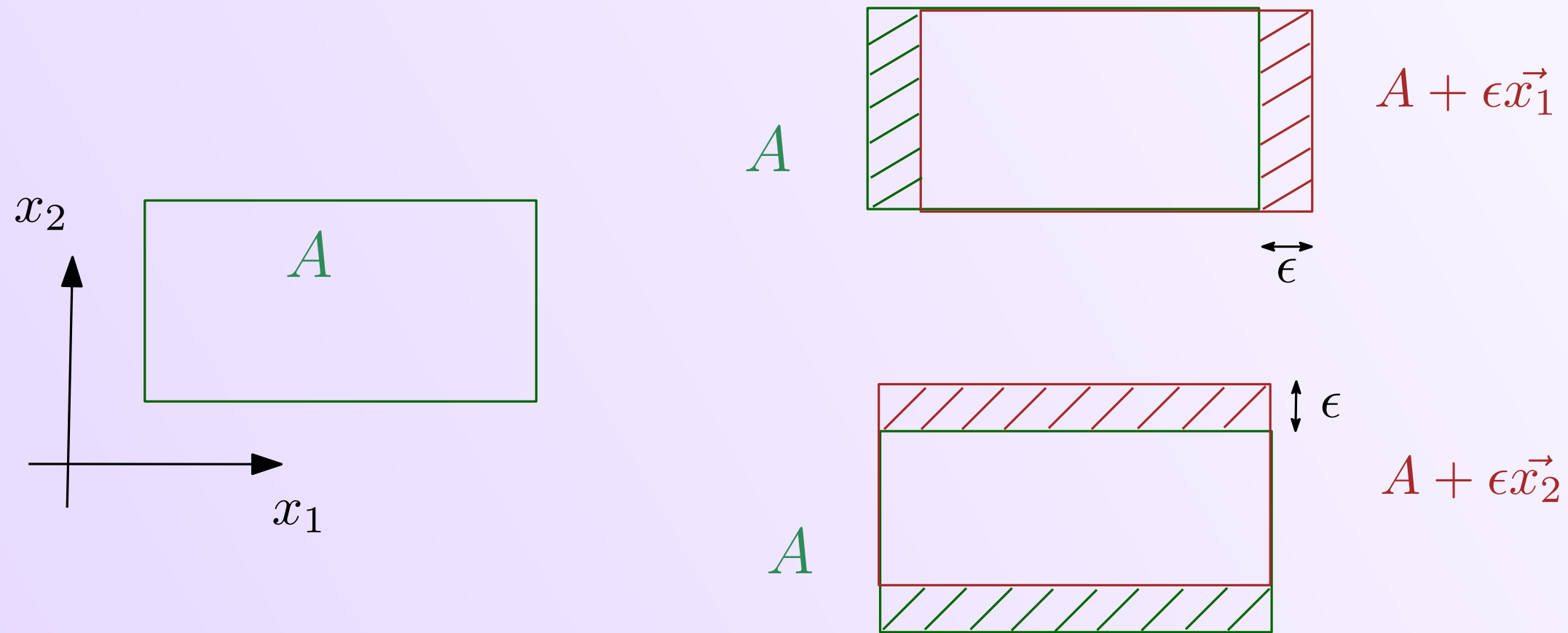
$$e^{2H(X)/n} + e^{2H(Y)/n} \leq e^{2H(X+Y)/n}$$

Isoperimetric inequality for entropies

$$\frac{1}{n} J(X) e^{2H(X)/n} \geq 2\pi e$$

$$\text{"length"} \quad \lim_{\epsilon \rightarrow 0} \frac{e^{2H(X + \sqrt{\epsilon}Z)/n} - e^{2H(X)/n}}{\epsilon}$$

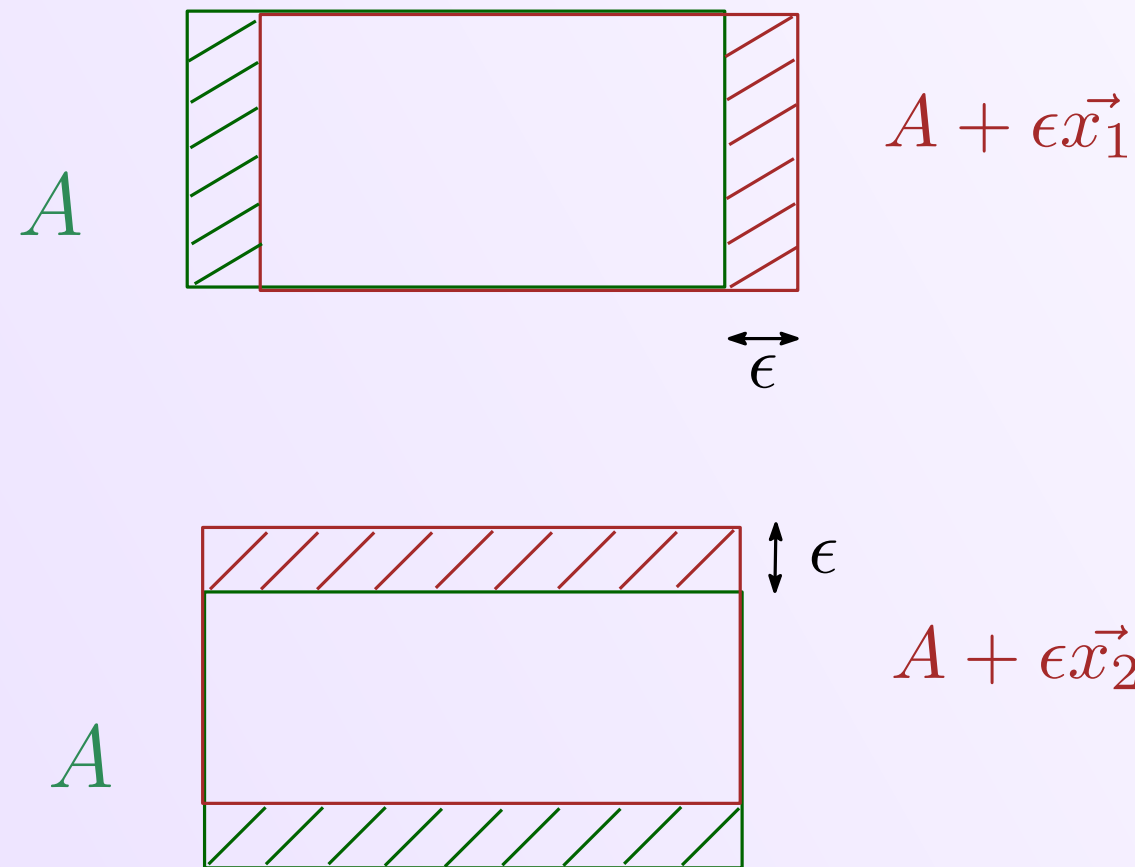
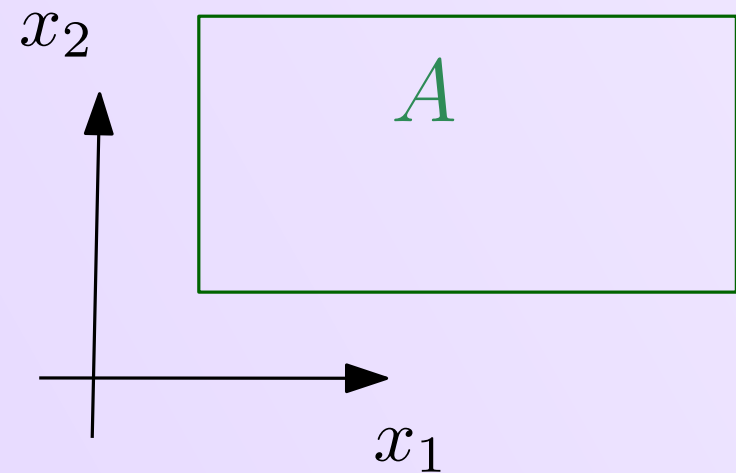
Geometric analogue of Fisher information



$D(A_1, A_2)$ = "difference" between A_1 and A_2

"Fisher Information" = $\frac{d^2}{d\epsilon^2} \Big|_{\epsilon=0} \sum_{j=1}^2 D(A, A + \epsilon \vec{x}_j)$

Geometric analogue of Fisher information



distribution translated in \vec{x}_j direction by ϵ

$D(A_1, A_2)$ = “difference” between A_1 and A_2

$$J(X) = \sum_{j=1}^n \left. \frac{\partial^2}{\partial \epsilon^2} \right|_{\epsilon=0} D(f || f^{\epsilon \vec{x}_j})$$

“Fisher Information” = $\left. \frac{d^2}{d\epsilon^2} \right|_{\epsilon=0} \sum_{j=1}^2 D(A, A + \epsilon \vec{x}_j)$

de Bruijn’s identity

$$= 2 \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} H(X + \sqrt{\epsilon} Z)$$

Geometry

Set $A \in \mathbb{R}^n$

volume $\text{vol}(A)$

addition $A + B$

Classical

Random variable X on \mathbb{R}^n
with prob. density function f_X

entropy power $e^{2H(X)/n}$

$$H(X) = - \int_{\mathbb{R}^n} f_X(x) \log f_X(x) dx$$

convolution: $X + Y$

$$f_{X+Y}(x) = \int f_X(x-z) f_Y(z) dz$$

Quantum

ρ - n -mode state.

$$[Q_j, P_k] = -[P_k, Q_j] = i\delta_{j,k}I$$

$$[Q_j, Q_k] = [P_j, P_k] = 0 \quad 1 \leq j, k, \leq n$$

$$N(\rho) = \exp\{S(\rho)/n\}$$

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$\rho_X + \rho_Y$??

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$$\rho_{X \boxplus_\lambda Y} = \text{Tr}_Y \left(U_\lambda (\rho_X \otimes \rho_Y) U_\lambda^\dagger \right)$$

beam splitter with transmissivity $\lambda \in [0, 1]$

$$U_{\lambda,1}^\dagger Q_{1(2)} U_{\lambda,1} = \sqrt{\lambda} Q_{1(2)} + \sqrt{1-\lambda} Q_{2(1)}$$

$$U_{\lambda,1}^\dagger P_{1(2)} U_{\lambda,1} = \sqrt{\lambda} P_{1(2)} + \sqrt{1-\lambda} P_{2(1)}$$

$$U_{\lambda,n} = U_{\lambda,1}^{\otimes n}$$

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$$\lambda e^{S(X)/n} + (1 - \lambda) e^{S(Y)/n} \leq e^{S(X \boxplus_\lambda Y)/n}$$

[Koenig, Smith 14] $\lambda = 1/2$

[de Palma et al 15]

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classical-quantum

$$f \star_t \rho = \int f(\xi) W(\sqrt{t}\xi) \rho W^\dagger(\sqrt{t}\xi) d\xi$$

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Classical-quantum entropy power inequality

$$t e^{H(f)/n} + e^{S(\rho)/n} \leq e^{S(f \star_t \rho)/n} \quad t \geq 0$$

Isoperimetric inequality for entropies

$$\frac{1}{n} J(\rho) e^{S(\rho)/n} \geq 4\pi e$$

Classical vs Quantum information theory

Classical

X - \mathbb{R}^n -valued r.v. with a prob. density function f_X .

Entropy

Shannon

$$H(X) = H(f) = - \int f(x) \log f(x) dx$$

Entropy Power $N(f) = \exp\{2H(f)/n\}$

Relative entropy $D(f||g) = \int f(x) \log \frac{f(x)}{g(x)} dx$

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von Neumann

$$S(\rho) = -\text{Tr}(\rho \log \rho)$$

$$N(\rho) = \exp\{S(\rho)/n\}$$

$$D(\rho||\sigma) = \text{Tr}(\rho \log \rho - \rho \log \sigma)$$

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Fisher information matrix For $\vec{\theta} \in \mathbb{R}^n$, define

$$f^{(\theta)}(x) = f(x - \theta)$$

$$J(f^{(\theta)})|_{\theta=\theta_0} = \left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \Big|_{\theta=\theta_0} D(f^{(\theta_0)} || f^{(\theta)}) \right)_{i,j=1}^{2n}$$

Fisher information

$$J(f) = \text{Tr} \left(J(f^{(\theta)})|_{\theta=\theta_0} \right)$$

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Quantum Fisher information

For $\vec{\theta} \in \mathbb{R}^{2n}$ the Weyl displacement operators are defined as

$$W(\vec{\theta}) = \exp\{i\sqrt{2\pi} (\theta_1 P_1 - \theta_2 Q_1 + \cdots + \theta_{2n-1} P_n - \theta_{2n} Q_n)\}$$

Translated state is

$$\rho^{(\theta)} = W(\theta)\rho W^\dagger(\theta)$$

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Weyl operators translate position and momentum operators

$$W(\vec{\theta})Q_jW^\dagger(\vec{\theta}) = Q_j + \theta I \quad W(\vec{\theta})P_jW^\dagger(\vec{\theta}) = P_j + \theta I$$

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Weyl operators translate position and momentum operators

$$W(\vec{\theta})Q_jW^\dagger(\vec{\theta}) = Q_j + \theta I \quad W(\vec{\theta})P_jW^\dagger(\vec{\theta}) = P_j + \theta I$$

Fisher information matrix

$$J(\rho^{(\theta)})|_{\theta=\theta_0} = \left(\frac{\partial^2}{\partial\theta_i\partial\theta_j} \Big|_{\theta=\theta_0} D(\rho^{(\theta_0)} || \rho^{(\theta)}) \right)_{i,j=1}^{2n}$$

Fisher information

$$J(\rho) = \text{Tr} \left(J(\rho^{(\theta)})|_{\theta=\theta_0} \right)$$

Inequalities

Classical

$$(f_X, f_Y) \rightarrow f_{X+Y}(z) = \int f_X(z-x)f_Y(x)dx$$

Quantum

$$(f, \rho) \rightarrow f \star_t \rho = \int f(\xi)W(\sqrt{t}\xi)\rho W^\dagger(\sqrt{t}\xi)d\xi$$

$t = 1$ [Werner '84]

Inequalities

Classical

$$(f_X, f_Y) \rightarrow f_{X+Y}(z) = \int f_X(z-x)f_Y(x)dx$$

- the Fisher information ineq.

for $\lambda \in [0, 1]$

$$J\left(\sqrt{\lambda}X + \sqrt{1-\lambda}Y\right) \leq \lambda J(X) + (1-\lambda)J(Y)$$

- Stam inequality

$$J(X+Y)^{-1} \geq J(X)^{-1} + J(Y)^{-1}$$

Quantum

$$(f, \rho) \rightarrow f \star_t \rho = \int f(\xi)W(\sqrt{t}\xi)\rho W^\dagger(\sqrt{t}\xi)d\xi$$

$t = 1$ [Werner '84]

- the Fisher information ineq.

for $\omega = \sqrt{t}\omega_c + \omega_q$

$$\omega^2 J(f \star_t \rho) \leq \omega_c^2 J(f) + \omega_q^2 J(\rho)$$

- Stam inequality

$$J(f \star_t \rho)^{-1} \geq tJ(f)^{-1} + J(\rho)^{-1}$$

Inequalities

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Z - Gaussian r.v. : $f_Z(\vec{\xi}) = (2\pi)^{-n/2}e^{-|\xi|^2/2}$

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Quantum Diffusion Semigroup

For Z - Gaussian r.v. : $f_Z(\vec{\xi}) = (2\pi)^{-n} e^{-|\xi|^2/2}$

$$f_Z \star_t \rho = \frac{1}{(2\pi)^n} \int e^{-\|\vec{\xi}\|^2/2} W(\sqrt{t}\vec{\xi}) \rho W^\dagger(\sqrt{t}\vec{\xi}) d\vec{\xi}$$

Quantum Diffusion Semigroup

For Z - Gaussian r.v. : $f_Z(\vec{\xi}) = (2\pi)^{-n} e^{-|\xi|^2/2}$

$$\begin{aligned} f_Z \star_t \rho &= \frac{1}{(2\pi)^n} \int e^{-\|\vec{\xi}\|^2/2} W(\sqrt{t}\vec{\xi}) \rho W^\dagger(\sqrt{t}\vec{\xi}) d\vec{\xi} \\ &= e^{t\mathcal{L}}(\rho) \end{aligned}$$

for a Liouvillean

$$\mathcal{L}(\rho) = -\pi \sum_{j=1}^{2n} [Q_j, [Q_j, \rho]] + [P_j, [P_j, \rho]]$$

The state satisfies

$$\rho(t) = e^{t\mathcal{L}}(\rho) : \quad \frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t))$$

Inequalities

Classical

- de Bruijn identity

$$J(X + \sqrt{t}Z) = 2 \frac{\partial}{\partial t} H(X + \sqrt{t}Z)$$

- Concavity of the entropy power

$$\left. \frac{d^2}{d\epsilon^2} \right|_{\epsilon=0} \{N(X + \sqrt{\epsilon}Z)\} \leq 0$$

- Fisher information isoperimetric inequality

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left\{ \left[\frac{1}{n} \{J(X + \sqrt{\epsilon}Z)\} \right]^{-1} \right\} \geq 1$$

- Entropy power inequality

$$N(X + Y) \geq N(X) + N(Y)$$

- Isoperimetric inequality

$$\frac{1}{n} J(X) N(X) \geq 2\pi e$$

Quantum

$$e^{t\mathcal{L}}(\rho) = f_Z \star_t \rho$$

$$J(\rho) = 2 \left. \frac{d}{dt} S(e^{t\mathcal{L}}(\rho)) \right|_{t=0} \quad [\text{Koenig, Smith '14}]$$

$$\left. \frac{d^2}{dt^2} \right|_{t=0} N(e^{t\mathcal{L}}(\rho)) \leq 0$$

$$\left. \frac{d}{dt} \right|_{t=0} \left\{ \left[\frac{1}{2n} J(e^{t\mathcal{L}}(\rho)) \right]^{-1} \right\} \geq 1$$

$$N(f \star_t \rho) \geq tN(f) + N(\rho)$$

$$\frac{1}{n} J(\rho) N(\rho) \geq 4\pi e$$

Quantum Inequalities

Rescaling

If f_X is a prob. density function of a r.v. X , then

$$f_X \star_t \rho = f_{\sqrt{t}X} \star_1 \rho$$

here the r.v. $\sqrt{t}X$ is given by $f_{\sqrt{t}X}(\zeta) = f(\zeta/\sqrt{t})/(\sqrt{t})^{2n}$

Addition

$$f_X \star_1 (f_Y \star_1 \rho) = f_{X+Y} \star_1 \rho$$

where $(f_X, f_Y) \rightarrow f_{X+Y}(z) = \int f_X(z-x)f_Y(x)dx$

Translation

Let $\omega_c, \omega_q > 0$, and $t \geq 0$. Then for all $\theta \in \mathbb{R}^{2n}$

$$(f \star_t \rho)^{(\omega\theta)} = f^{(\omega_c\theta)} \star_t \rho^{(\omega_q\theta)}$$

$$\omega = \sqrt{t}\omega_c + \omega_q$$

Quantum Inequalities

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Let $\omega_c, \omega_q > 0$, and $t \geq 0$. Then for all $\theta \in \mathbb{R}^{2n}$

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$$\omega = \sqrt{t}\omega_c + \omega_q$$

Data Processing Inequality

$$D(f \star_t \rho \| g \star_t \sigma) \leq D(f \| g) + D(\rho \| \sigma)$$

Stam Inequality

Theorem

Let $\omega_c, \omega_q \in \mathbf{R}$, and $t \geq 0$. Then

$$\omega^2 J(f \star_t \rho) \leq \omega_c^2 J(f) + \omega_q^2 J(\rho) \quad \text{for } \omega = \sqrt{t}\omega_c + \omega_q$$

In particular,

$$J(f \star_t \rho)^{-1} - J(\rho)^{-1} - tJ(f)^{-1} \geq 0$$

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Let $\omega_c, \omega_q \in \mathbf{R}$, and $t \geq 0$. Then

$$\omega^2 J(f \star_t \rho) \leq \omega_c^2 J(f) + \omega_q^2 J(\rho) \quad \text{for } \omega = \sqrt{t}\omega_c + \omega_q$$

In particular,

$$J(f \star_t \rho)^{-1} - J(\rho)^{-1} - tJ(f)^{-1} \geq 0$$

Proof

By Data processing inequality

$$\text{Tr} \left(J(f^{(\omega_c \vec{\theta})} \star_t \rho^{(\omega_q \vec{\theta})}); \vec{\theta} \right) \Big|_{\vec{\theta}=\vec{\theta}_0} \leq \text{Tr} \left(J(f^{(\omega_c \vec{\theta})}); \vec{\theta} \right) \Big|_{\vec{\theta}=\vec{\theta}_0} + \text{Tr} \left(J(\rho^{(\omega_q \vec{\theta})}); \vec{\theta} \right) \Big|_{\vec{\theta}=\vec{\theta}_0}$$

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Theorem

Let $\omega_c, \omega_q \in \mathbf{R}$, and $t \geq 0$. Then

$$\omega^2 J(f \star_t \rho) \leq \omega_c^2 J(f) + \omega_q^2 J(\rho) \quad \text{for } \omega = \sqrt{t}\omega_c + \omega_q$$

In particular,

$$J(f \star_t \rho)^{-1} - J(\rho)^{-1} - tJ(f)^{-1} \geq 0$$

Proof

By Data processing inequality

$$\begin{aligned} \text{Tr} \left(J(f^{(\omega_c \vec{\theta})} \star_t \rho^{(\omega_q \vec{\theta})}); \vec{\theta} \right) \Big|_{\vec{\theta}=\vec{\theta}_0} &\leq \text{Tr} \left(J(f^{(\omega_c \vec{\theta})}); \vec{\theta} \right) \Big|_{\vec{\theta}=\vec{\theta}_0} + \text{Tr} \left(J(\rho^{(\omega_q \vec{\theta})}); \vec{\theta} \right) \Big|_{\vec{\theta}=\vec{\theta}_0} \\ &\leftarrow \\ \text{Tr} \left(J(\{f \star_t \rho\}^{(\omega \vec{\theta})}); \vec{\theta} \right) \Big|_{\vec{\theta}=\vec{\theta}_0} \end{aligned}$$

Stam Inequality

Theorem

Let $\omega_c, \omega_q \in \mathbf{R}$, and $t \geq 0$. Then

$$\omega^2 J(f \star_t \rho) \leq \omega_c^2 J(f) + \omega_q^2 J(\rho) \quad \text{for } \omega = \sqrt{t}\omega_c + \omega_q$$

In particular,

$$J(f \star_t \rho)^{-1} - J(\rho)^{-1} - tJ(f)^{-1} \geq 0$$

Proof

By Data processing inequality

$$\text{Tr} \left(J(f^{(\omega_c \vec{\theta})} \star_t \rho^{(\omega_q \vec{\theta})}); \vec{\theta} \middle|_{\vec{\theta}=\vec{\theta}_0} \right) \leq \text{Tr} \left(J(f^{(\omega_c \vec{\theta})}); \vec{\theta} \middle|_{\vec{\theta}=\vec{\theta}_0} \right) + \text{Tr} \left(J(\rho^{(\omega_q \vec{\theta})}); \vec{\theta} \middle|_{\vec{\theta}=\vec{\theta}_0} \right)$$

$$\text{Tr} \left(J(\{f \star_t \rho\}^{(\omega \vec{\theta})}); \vec{\theta} \middle|_{\vec{\theta}=\vec{\theta}_0} \right)$$

$$\omega^2 J(f \star_t \rho)$$

Stam Inequality

Theorem

Let $\omega_c, \omega_q \in \mathbf{R}$, and $t \geq 0$. Then

$$\omega^2 J(f \star_t \rho) \leq \omega_c^2 J(f) + \omega_q^2 J(\rho) \quad \text{for } \omega = \sqrt{t}\omega_c + \omega_q$$

In particular,

$$J(f \star_t \rho)^{-1} - J(\rho)^{-1} - tJ(f)^{-1} \geq 0$$

Proof

By Data processing inequality

$$\text{Tr} \left(J(f^{(\omega_c \vec{\theta})} \star_t \rho^{(\omega_q \vec{\theta})}); \vec{\theta} \right) \Big|_{\vec{\theta}=\vec{\theta}_0} \leq \text{Tr} \left(J(f^{(\omega_c \vec{\theta})}); \vec{\theta} \right) \Big|_{\vec{\theta}=\vec{\theta}_0} + \text{Tr} \left(J(\rho^{(\omega_q \vec{\theta})}); \vec{\theta} \right) \Big|_{\vec{\theta}=\vec{\theta}_0}$$

$$\text{Tr} \left(J(\{f \star_t \rho\}^{(\omega \vec{\theta})}); \vec{\theta} \right) \Big|_{\vec{\theta}=\vec{\theta}_0}$$

$$\omega^2 J(f \star_t \rho)$$

$$J(f^{(\omega_c \vec{\theta})}; \vec{\theta}) = \omega_c^2 J(f^{(\vec{\theta})}; \vec{\theta})$$

$$\omega_c^2 J(f)$$

Stam Inequality

Theorem

Let $\omega_c, \omega_q \in \mathbf{R}$, and $t \geq 0$. Then

$$\omega^2 J(f \star_t \rho) \leq \omega_c^2 J(f) + \omega_q^2 J(\rho) \quad \text{for } \omega = \sqrt{t}\omega_c + \omega_q$$

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$$\text{Tr} \left(J(\{f \star_t \rho\}^{(\omega \vec{\theta})}); \vec{\theta} \right) \Big|_{\vec{\theta}=\vec{\theta}_0}$$

$$\omega^2 J(f \star_t \rho)$$

$$J(f^{(\omega_c \vec{\theta})}); \vec{\theta} = \omega_c^2 J(f^{(\vec{\theta})}); \vec{\theta}$$

$$\omega_c^2 J(f)$$

$$J(\rho^{(\omega_q \vec{\theta})}); \vec{\theta} = \omega_q^2 J(\rho^{(\vec{\theta})}); \vec{\theta}$$

$$\omega_q^2 J(\rho)$$

Stam Inequality

Theorem

Let $\omega_c, \omega_q \in \mathbf{R}$, and $t \geq 0$. Then

$$\omega^2 J(f \star_t \rho) \leq \omega_c^2 J(f) + \omega_q^2 J(\rho) \quad \text{for } \omega = \sqrt{t}\omega_c + \omega_q$$

In particular,

$$J(f \star_t \rho)^{-1} - J(\rho)^{-1} - tJ(f)^{-1} \geq 0$$

Proof

By Data processing inequality

$$\begin{aligned} \text{Tr} \left(J(f^{(\omega_c \vec{\theta})} \star_t \rho^{(\omega_q \vec{\theta})}); \vec{\theta} \right) \Big|_{\vec{\theta}=\vec{\theta}_0} &\leq \text{Tr} \left(J(f^{(\omega_c \vec{\theta})}); \vec{\theta} \right) \Big|_{\vec{\theta}=\vec{\theta}_0} + \text{Tr} \left(J(\rho^{(\omega_q \vec{\theta})}); \vec{\theta} \right) \Big|_{\vec{\theta}=\vec{\theta}_0} \\ \text{Tr} \left(J(\{f \star_t \rho\}^{(\omega \vec{\theta})}); \vec{\theta} \right) \Big|_{\vec{\theta}=\vec{0}} &\quad \begin{array}{c} \swarrow \\ J(f^{(\omega_c \vec{\theta})}; \vec{\theta}) = \omega_c^2 J(f^{(\vec{\theta})}; \vec{\theta}) \\ \downarrow \\ \omega_c^2 J(f) \end{array} \quad \begin{array}{c} \downarrow \\ J(\rho^{(\omega_q \vec{\theta})}; \vec{\theta}) = \omega_q^2 J(\rho^{(\vec{\theta})}; \vec{\theta}) \\ \downarrow \\ \omega_q^2 J(\rho) \end{array} \end{aligned}$$

Taking $\omega_c = \frac{\sqrt{t}J(f)^{-1}}{J(\rho)^{-1}+tJ(f)^{-1}}$ and $\omega_q = \frac{J(\rho)^{-1}}{J(\rho)^{-1}+tJ(f)^{-1}}$, we obtain

$$J(f \star_t \rho)^{-1} - J(\rho)^{-1} - tJ(f)^{-1} \geq 0$$

□

Concavity of entropy power of heat diffusion semigroup

Classical heat diffusion semigroup

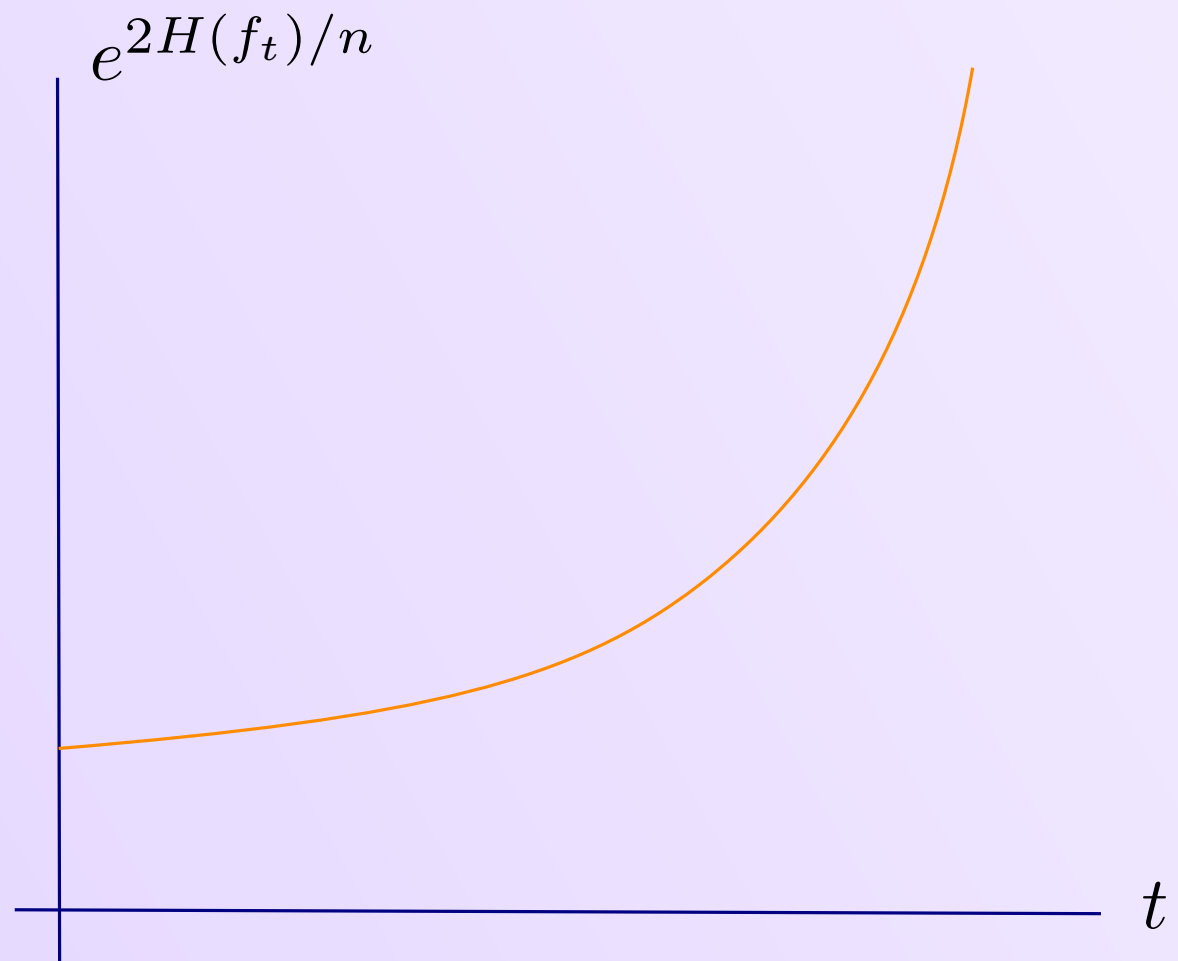
$$\frac{\partial}{\partial t} f_t = \Delta f_t \quad \text{where } \Delta = \sum_j \frac{\partial^2}{\partial x_j^2}$$

Concavity of entropy power of heat diffusion semigroup

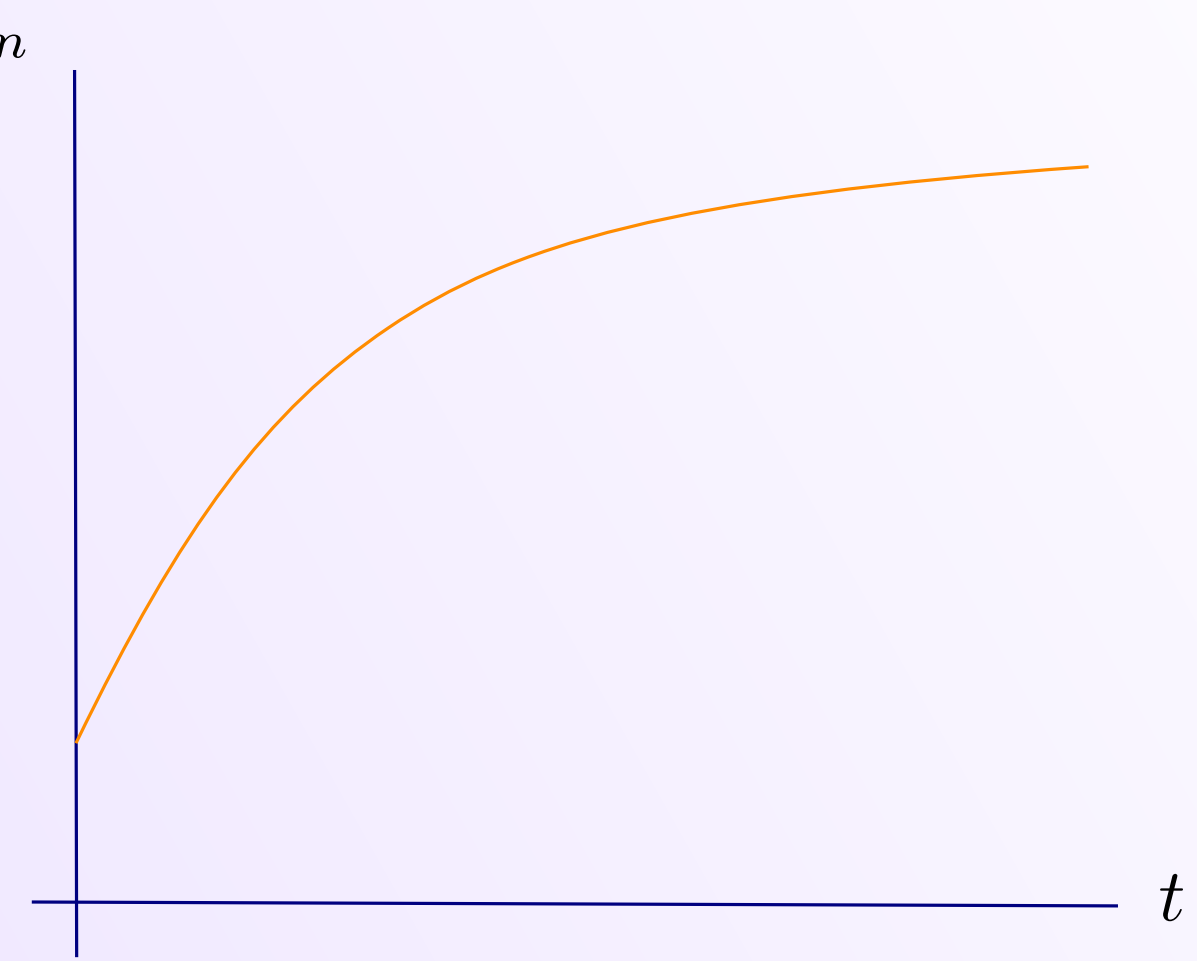
Classical heat diffusion semigroup

$$\frac{\partial}{\partial t} f_t = \Delta f_t$$

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or



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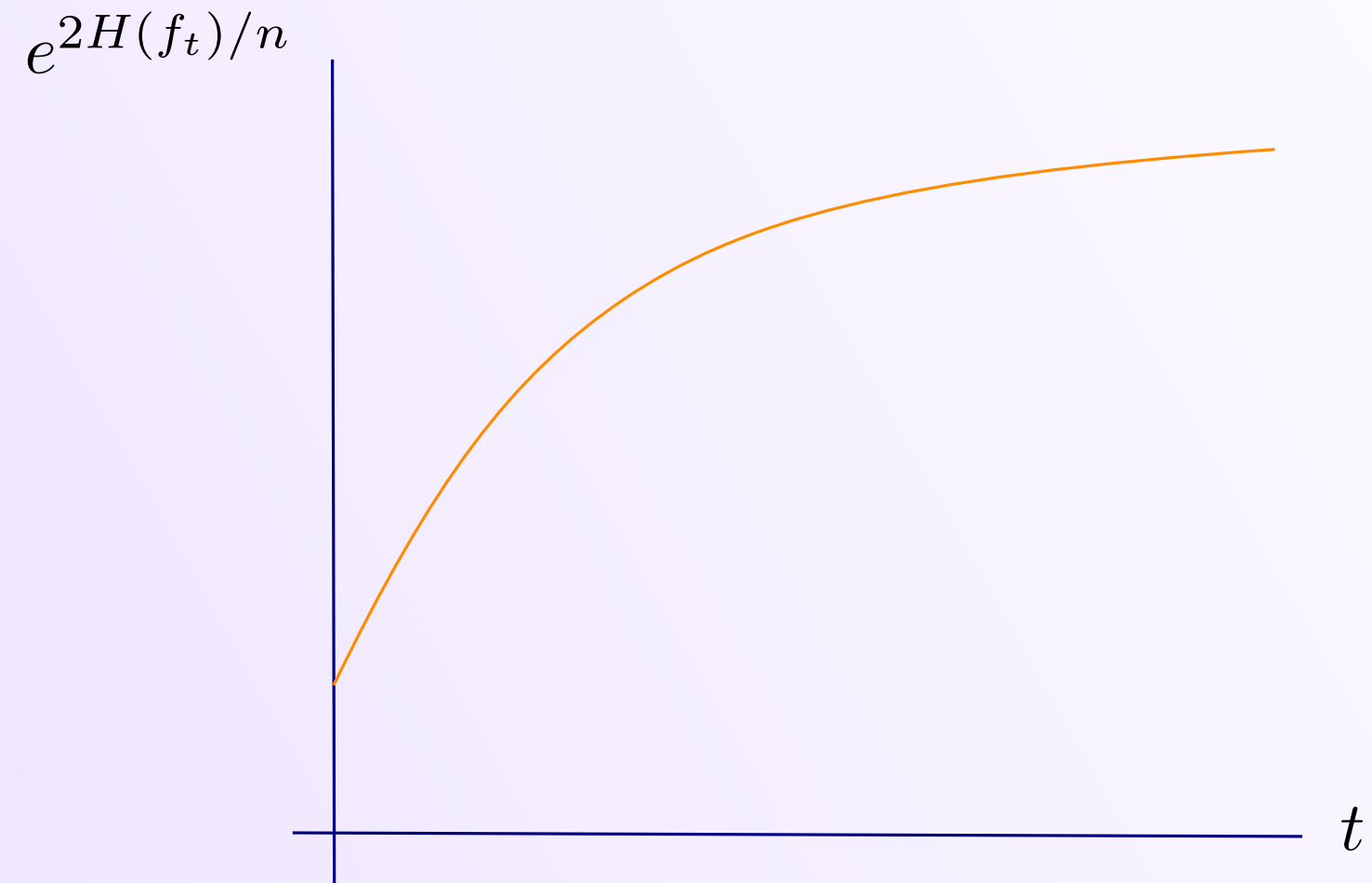
$$\text{where } \Delta = \sum_j \frac{\partial^2}{\partial x_j^2}$$

Theorem

$$\frac{d^2}{dt^2} e^{2H(f_t)/n} \leq 0$$

Dembo, Thomas, Cover '91

Villan '00



Concavity of entropy power of diffusion semigroup

Quantum diffusion semigroup

$$\frac{d}{dt}\rho_t = \mathcal{L}(\rho_t) \quad \text{with } \mathcal{L}(\rho) = -\pi \sum_{j=1}^{2n} [Q_j, [Q_j, \rho]] + [P_j, [P_j, \rho]]$$

Concavity of entropy power of diffusion semigroup

Quantum diffusion semigroup

$$\frac{d}{dt}\rho_t = \mathcal{L}(\rho_t) \quad \text{with } \mathcal{L}(\rho) = -\pi \sum_{j=1}^{2n} [Q_j, [Q_j, \rho]] + [P_j, [P_j, \rho]]$$

Theorem

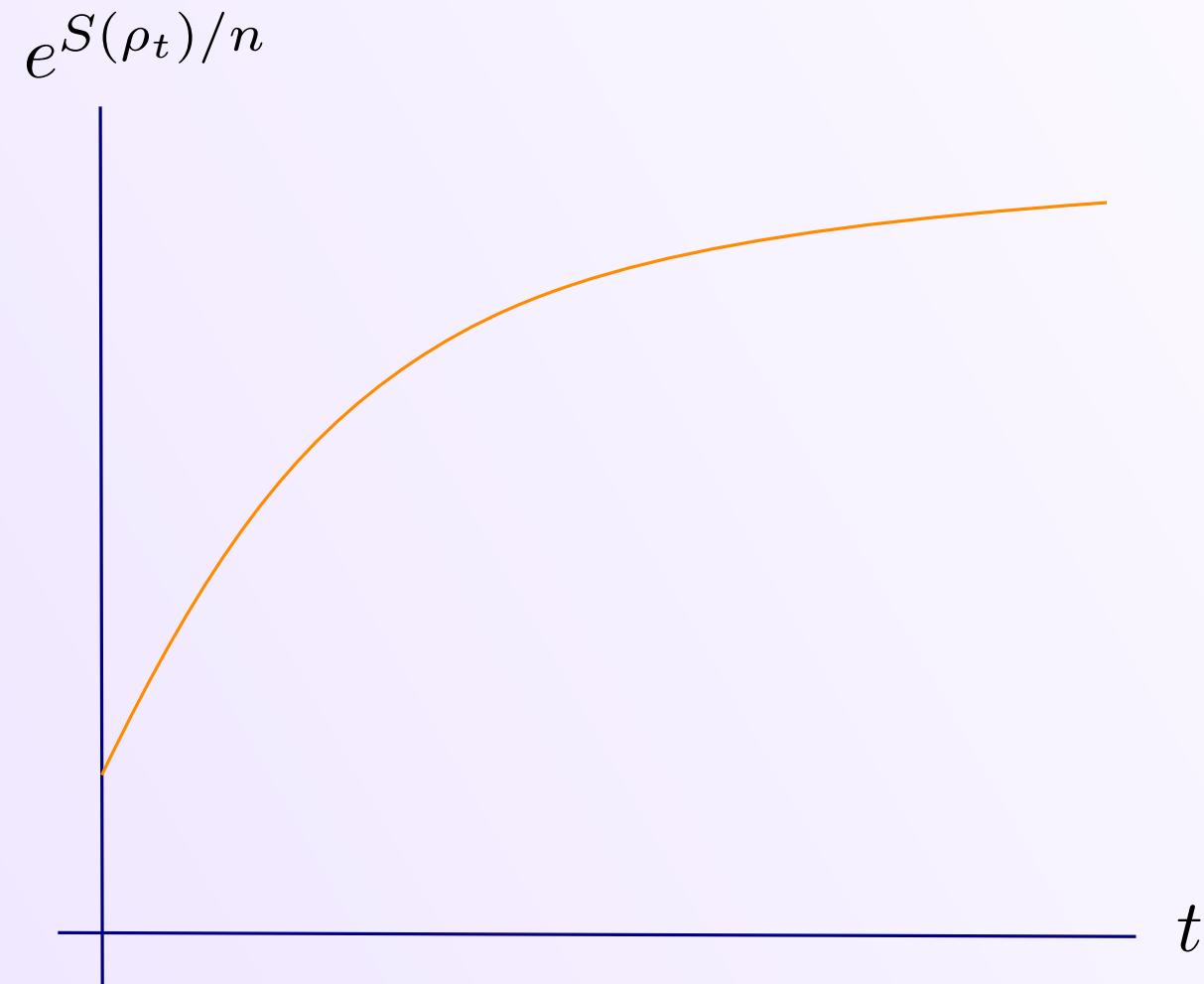
$$\left. \frac{d^2}{dt^2} \right|_{t=0} N(e^{t\mathcal{L}}(\rho)) \leq 0$$

Recall that for a Gaussian r.v. Z we have

$$e^{t\mathcal{L}}(\rho) = f_Z \star_t \rho$$

Therefore,

$$\left. \frac{d^2}{dt^2} \right|_{t=0} N(f_Z \star_t \rho) \leq 0$$



Quantum Fisher information isoperimetric inequality

$$\frac{d}{dt} \Big|_{t=0} \left\{ \left[\frac{1}{2n} J(e^{t\mathcal{L}}(\rho)) \right]^{-1} \right\} \geq 1$$

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$$\frac{d}{dt} \Big|_{t=0} \left\{ \left[\frac{1}{2n} J(f_Z \star_t \rho) \right]^{-1} \right\} \geq 1$$

Proof: Take $f = f_Z$ in q. Stam inequality:

$$\frac{1}{t} \left\{ J(f_Z \star_t \rho)^{-1} - J(\rho)^{-1} \right\} \geq J(f_Z)^{-1} = (2n)^{-1}$$

Take a limit $t \rightarrow \infty$.

□

Entropy power inequality

$$N(f \star_t \rho) \leq tN(f) + N(\rho)$$

$$e^{S(f \star_t \rho)/n} \leq te^{2H(f)/n} + e^{S(\rho)/n}$$

Entropy power inequality

$$N(f \star_t \rho) \leq tN(f) + N(\rho)$$

$$e^{S(f \star_t \rho)/n} \leq te^{2H(f)/n} + e^{S(\rho)/n}$$

In particular,

$$N(e^{t\mathcal{L}}(\rho)) \geq N(\rho) + t 2\pi e$$

Isoperimetric Inequality for Entropies

$$\frac{1}{n} J(\rho) N(\rho) \geq 4\pi e$$

Isoperimetric Inequality for Entropies

$$\frac{1}{n} J(\rho) N(\rho) \geq 4\pi e$$

Proof:

from de Bruijn identity:

$$\left. \frac{d}{dt} N(\rho(t)) \right|_{t=0} = \frac{1}{2n} J(\rho) N(\rho)$$

denoting $\rho(t) = e^{t\mathcal{L}}(\rho)$, $t \geq 0$, the entropy power inequality yields:

$$\frac{1}{t} [N(\rho(t)) - N(\rho)] \geq 2\pi e$$

take a limit $t \rightarrow \infty$.

□

Isoperimetric Inequality for Entropies

$$\frac{1}{n} J(\rho) N(\rho) \geq 4\pi e$$

Optimality: let $\rho = \omega_{\mathbf{n}}$ be a Gaussian thermal state with a mean photon number \mathbf{n}

Then $S(\omega_{\mathbf{n}}) = g(\mathbf{n}) = (\mathbf{n} + 1) \log(\mathbf{n} + 1) - \mathbf{n} \log \mathbf{n}$

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Under \mathcal{L} the state evolves as $e^{t\mathcal{L}}(\omega_{\mathbf{n}}) = \omega_{\mathbf{n}_t}$ where $\mathbf{n}_t = \mathbf{n} + 2\pi t$

In particular, by de Bruijn identity

$$J(\omega_{\mathbf{n}}) = 2 \left. \frac{d}{dt} \right|_{t=0} S(e^{t\mathcal{L}}(\omega_{\mathbf{n}})) = 4\pi \log\left(\frac{\mathbf{n}+1}{\mathbf{n}}\right)$$

Isoperimetric Inequality for Entropies

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In particular, by de Bruijn identity

$$J(\omega_{\mathbf{n}}) = 2 \left. \frac{d}{dt} \right|_{t=0} S(e^{t\mathcal{L}}(\omega_{\mathbf{n}})) = 4\pi \log\left(\frac{\mathbf{n}+1}{\mathbf{n}}\right)$$

Also

$$N(\omega_{\mathbf{n}}) = \exp(S(\omega_{\mathbf{n}})/1) = \frac{(\mathbf{n}+1)^{\mathbf{n}+1}}{\mathbf{n}^{\mathbf{n}}}$$

Therefore

$$J(\omega_{\mathbf{n}})N(\omega_{\mathbf{n}}) = 4\pi \left(\frac{\mathbf{n}+1}{\mathbf{n}}\right)^{\mathbf{n}} \log\left(\frac{\mathbf{n}+1}{\mathbf{n}}\right)^{\mathbf{n}+1} \xrightarrow{\mathbf{n} \rightarrow \infty} 4\pi e$$

Independent results by Rouzé, Datta, Pautrat

Quantum Nash Inequality

For n -mode Gaussian states $\{\rho_j\}_1^N$, and $\lambda_j \in \mathbb{C}$, let $A = \sum_j \lambda_j \rho_j$

Then

$$\|A\|_2^{2+2/n} \leq -C \|A\|_1^{2/n} \text{Tr}(A^* \mathcal{L}(A))$$

Non-commutative ultracontractivity

There exists $\kappa_n > 0$, s.t. for any $t > 0$:

$$\|e^{t\mathcal{L}}\|_{1 \rightarrow 2} \leq \kappa_n t^{-n/2}$$

$$\text{here } \|\Lambda\|_{p \rightarrow q} = \sup_{\|A\|_p=1} \|\Lambda(A)\|_q$$

$$\|e^{t\mathcal{L}}\|_{1 \rightarrow \infty} \leq 2^n \kappa_n^2 t^{-n}$$

$$\|A\|_p^p = \text{Tr}(|A|^p)$$

Therefore,

$$S(e^{t\mathcal{L}}(\rho)) \geq n \log(2\pi \kappa_n^{-\frac{2}{n}} t)$$

$$\text{Tr}(\{e^{t\mathcal{L}}(\rho)\}^2) \leq \kappa_n^2 t^{-n}$$

Quantum Blachman-Stam inequality

For any $\alpha, \beta > 0$ and $t > 0$,

$$(\alpha + \beta)^2 J(e^{t\mathcal{L}}(\rho)) \leq \alpha^2 J(\rho) + \frac{2n\beta^2}{t}$$

Follows directly from Stam Inequality

Quantum Ornstein-Uhlenbeck semigroup

$$n = 1$$

Quantum Attenuator:

$$\mathcal{L}_-(\rho) = a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}$$

Quantum Amplifier:

$$\mathcal{L}_+(\rho) = a^\dagger \rho a - \frac{1}{2}\{aa^\dagger, \rho\}$$

Let $\rho = \omega_{\mathbf{n}}$ be a Gaussian thermal state with a mean photon number \mathbf{n} . Then $\rho_{\pm}(t) = \omega_{\mathbf{n}_{\pm}(t)}$

Quantum Ornstein-Uhlenbeck semigroup

$$n = 1$$

Quantum Attenuator:

$$\mathcal{L}_-(\rho) = a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}$$

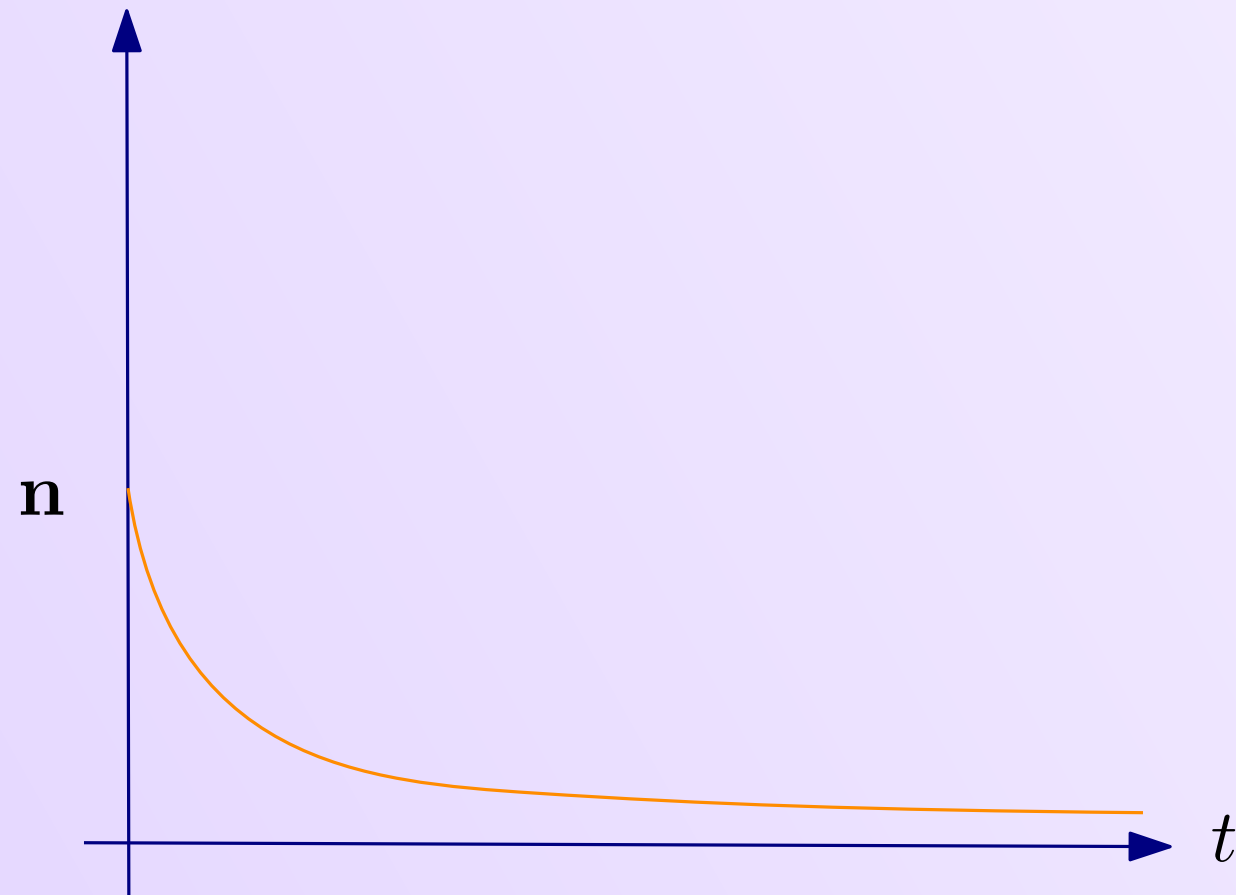
Quantum Amplifier:

$$\mathcal{L}_+(\rho) = a^\dagger \rho a - \frac{1}{2}\{aa^\dagger, \rho\}$$

Let $\rho = \omega_{\mathbf{n}}$ be a Gaussian thermal state with a mean photon number \mathbf{n} . Then $\rho_{\pm}(t) = \omega_{\mathbf{n}_{\pm}(t)}$

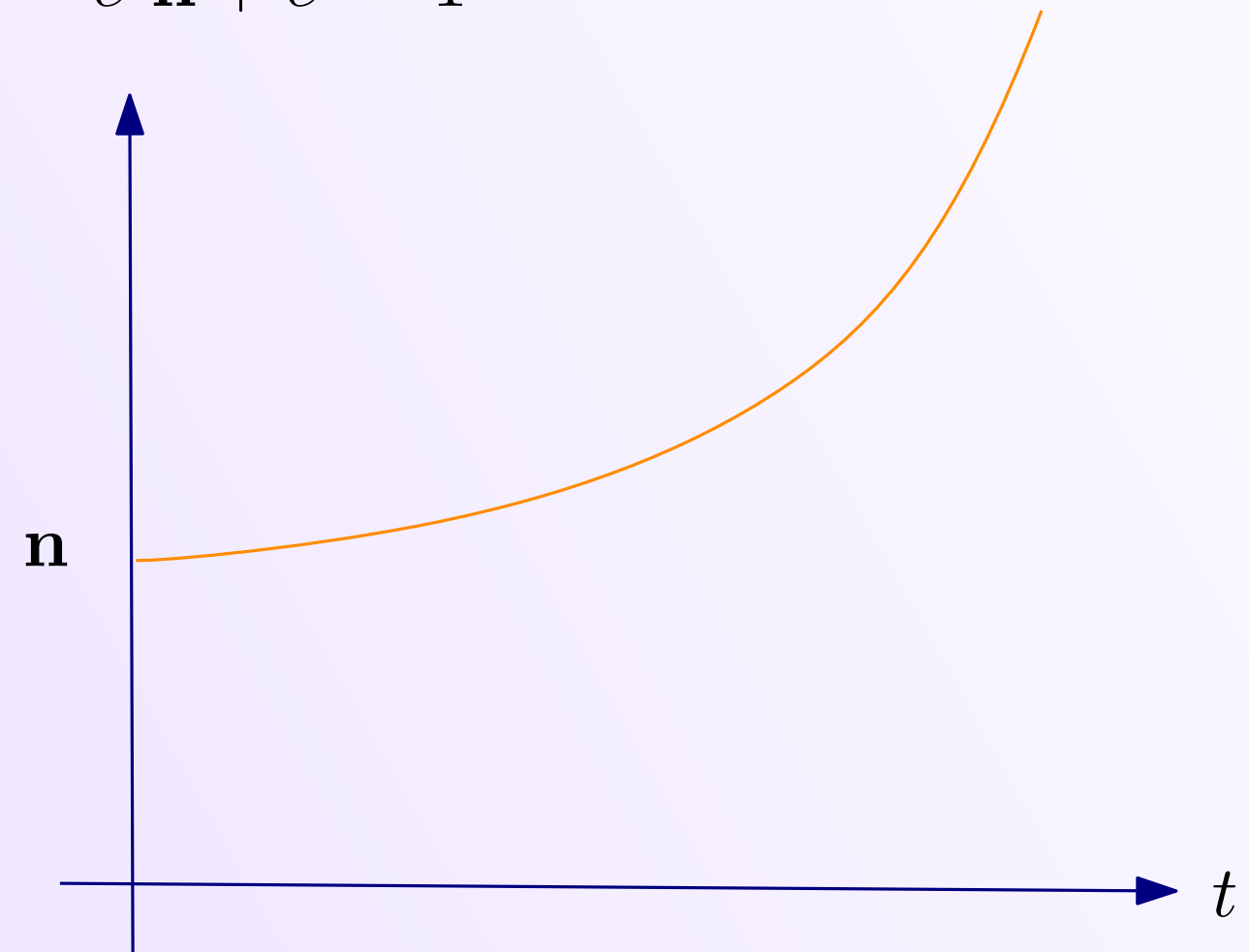
Under \mathcal{L}_-

$$\mathbf{n}_-(t) = e^{-t}\mathbf{n}$$



Under \mathcal{L}_+

$$\mathbf{n}_+(t) = e^t\mathbf{n} + e^t - 1$$



Quantum Ornstein-Uhlebeck semigroup

One-parameter group of CPTP maps $\{e^{\mathcal{L}_{\mu,\lambda}}\}_{t \geq 0}$, generated by

$$\mathcal{L}_{\mu,\lambda} = \mu^2 \mathcal{L}_- + \lambda^2 \mathcal{L}_+ \quad \text{for } \mu > \lambda > 0$$

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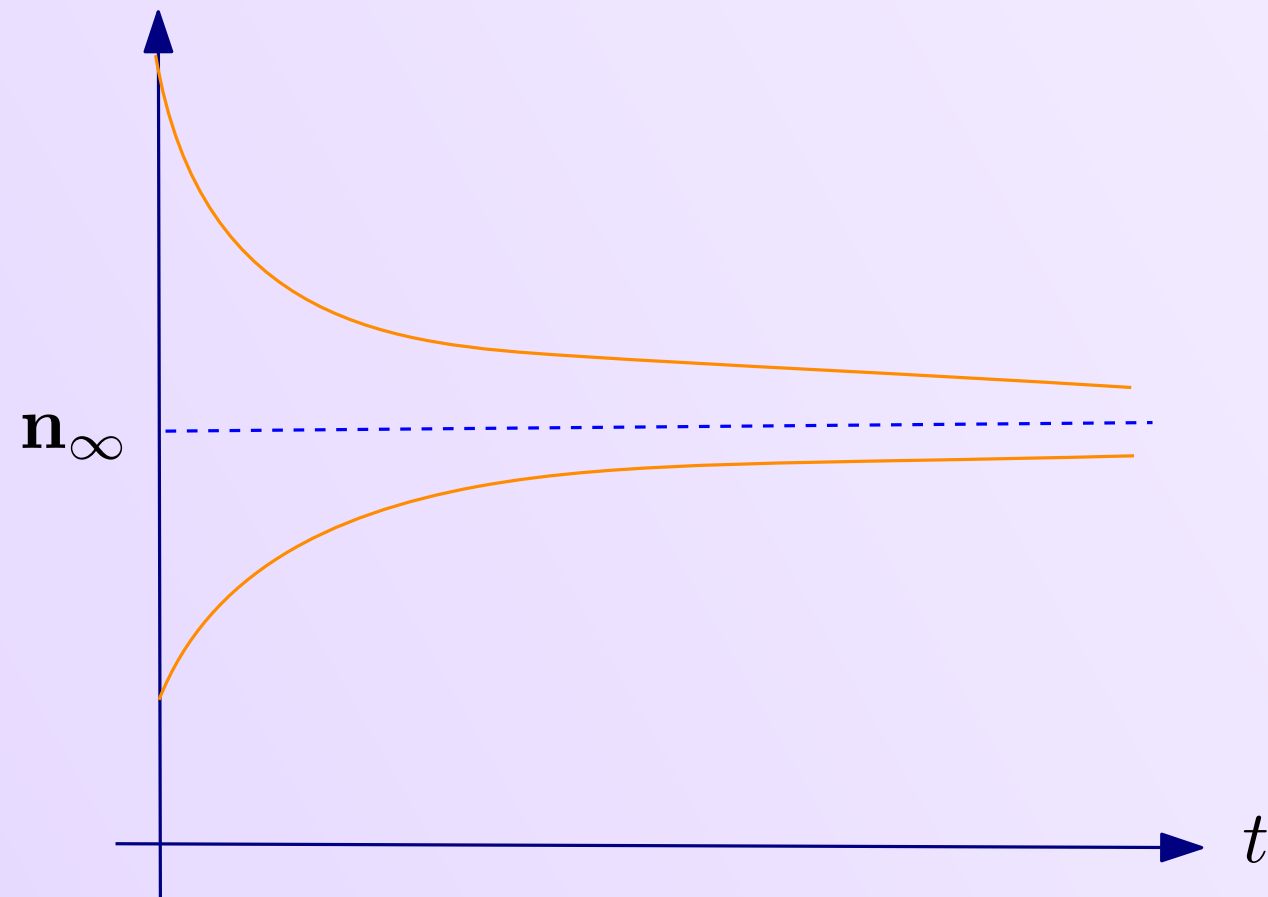
One-parameter group of CPTP maps $\{e^{\mathcal{L}_{\mu,\lambda}t}\}_{t \geq 0}$, generated by

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Then $\omega_{\mathbf{n}(t)}$ is s.t.

$$\mathbf{n}(t) = e^{-(\mu^2 - \lambda^2)t} \mathbf{n} + (1 - e^{-(\mu^2 - \lambda^2)t}) \mathbf{n}_{\infty} \quad \text{with } \mathbf{n}_{\infty} = \text{tr}(\sigma_{\mu,\lambda} \hat{n}) = \frac{\lambda^2}{\mu^2 - \lambda^2}$$



Fast convergence of qOU semigroup

Theorem

For any one-mode Gaussian state ρ we have

$$D(e^{t\mathcal{L}_{\mu,\lambda}}(\rho)\|\sigma_{\mu,\lambda}) \leq e^{-(\mu^2-\lambda^2)t} D(\rho\|\sigma_{\mu,\lambda}) \quad \text{for all } t \geq 0$$

In other words,

$$\left. \frac{d}{dt} \right|_{t=0} D(e^{t\mathcal{L}_{\mu,\lambda}}(\rho)\|\sigma_{\mu,\lambda}) \leq -(\mu^2 - \lambda^2) D(\rho\|\sigma_{\mu,\lambda})$$

Moreover, for any $\zeta > \mu^2 - \lambda^2$ there exists a Gaussian state ρ s.t.

$$\left. \frac{d}{dt} \right|_{t=0} D(e^{t\mathcal{L}_{\mu,\lambda}}(\rho)\|\sigma_{\mu,\lambda}) > -\zeta D(\rho\|\sigma_{\mu,\lambda})$$

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Conjecture (June '16)

For any state ρ we have

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Proved by Calren, Mass in Sep '16

Quantum Log-Sobolev inequality

From Isoperimetric inequality for entropies for $A > 0$

$$-S(\rho) \leq AJ(\rho) - (2 + \log(4\pi A))$$

Quantum Log-Sobolev inequality

From Isoperimetric inequality for entropies

for $A > 0$

$$-S(\rho) \leq AJ(\rho) - (2 + \log(4\pi A))$$

$$-S(\rho) \leq \log \left\{ \frac{1}{4\pi e} J(\rho) \right\}$$

$$\begin{aligned} \log \left(\frac{1}{4\pi e} J(\rho) \right) &= \log \left(\frac{AJ(\rho)}{4\pi e A} \right) \\ &= \log \left(\frac{1}{4\pi e A} \right) + \log \left(AJ(\rho) \right) \end{aligned}$$

using $\log x \leq x - 1$

$$\leq AJ(\rho) - 2 - \log(4\pi A)$$

Quantum Log-Sobolev inequality

From Isoperimetric inequality for entropies for $A > 0$

$$-S(\rho) \leq AJ(\rho) - (2 + \log(4\pi A))$$

Note that

$$D(\rho \parallel \sigma_{\mu, \lambda}) = -S(\rho) - (\log \nu) \mathbf{n} - \log(1 - \nu) \quad \nu = \lambda^2 / \mu^2 < 1$$

Therefore

$$D(\rho \parallel \sigma_{\mu, \lambda}) \leq AJ(\rho) - 2 - \log(4\pi A) + \mathbf{n} \log \frac{1}{\nu} - \log(1 - \nu)$$

Classical Log-Sobolev inequality

[Carlen '91]

Gross' logarithmic Sobolev inequality is

$$\int |f|^2 \log |f|^2 e^{-\pi|x|^2} dx \leq \frac{1}{\pi} \int |\nabla f|^2 e^{-\pi|x|^2} dx \quad \text{for } \int |f|^2 e^{-\pi|x|^2} dx = 1$$

Let $g(x) = f(x)e^{-\pi|x|^2/2}$. Then

$$\int |g|^2 \log |g|^2 dx \leq \frac{1}{\pi} \int |\nabla g|^2 dx - n \quad \text{with } \int |g|^2 dx = 1$$

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Let r.v. X has density $h_X(x) = |g(x)|^2$ and r.v. Y has density $h_Y(x) = e^{-\pi|x|^2}$. Then

$$H(X|Y)$$


$$\int h_X \log h_X dx - \int h_X \log h_Y dx$$

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Quantum case:

$$D(\rho || \sigma_{\mu, \lambda}) \leq AJ(\rho) - 2 - \log(1 - \nu) - \log(4\pi A) + \mathbf{n} \log \frac{1}{\nu} \quad \begin{aligned} \nu &= \lambda^2 / \mu^2 < 1 \\ A &> 0 \end{aligned}$$

Fast convergence of Ornstein-Uhlenbeck semigroup

$$-\frac{d}{dt} \Big|_{t=0} D(e^{t\mathcal{L}_{\mu,\lambda}}(\rho) \|\sigma_{\mu,\lambda}) - (\mu^2 - \lambda^2) D(\rho \|\sigma_{\mu,\lambda}) \geq 0$$

Fast convergence of Ornstein-Uhlenbeck semigroup

$$-\frac{d}{dt} \Big|_{t=0} D(e^{t\mathcal{L}_{\mu,\lambda}}(\rho) \parallel \sigma_{\mu,\lambda}) - (\mu^2 - \lambda^2) D(\rho \parallel \sigma_{\mu,\lambda}) \geq 0$$

$$J_{\pm}(\rho) := 2 \frac{d}{dt} S(e^{t\mathcal{L}_{\pm}}(\rho))$$



$$\frac{\mu^2}{2} J_{-}(\rho) + \frac{\lambda^2}{2} J_{+}(\rho) - (\log \nu)(\mu^2 - \lambda^2) \mathbf{n} + \lambda^2 \log \nu$$

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[de Palma et al. '16]:

$$J_{-}(\rho) \geq f(S(\rho))$$

[Buscemi et al. 16]:

$$J_{+}(\rho) \geq 2$$

For large $S(\rho)$, e.g. $\mu^2 = 2, \lambda^2 = 1$: $S(\rho) \gtrsim 0.5$

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isoperimetric ineq.

$$\leq AJ(\rho) - 2 - \log(4\pi A) - \mathbf{n} \log \nu - \log(1 - \nu)$$

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Take $A = \frac{\lambda^2}{4\pi(\mu^2 - \lambda^2)}$

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Gaussian optimality:

$$\geq J_{-}(\omega_{\mathbf{n}}) = -2\mathbf{n} \log(1 + 1/\mathbf{n})$$

For small $\mathbf{n} = \text{Tr}(\rho \hat{n})$, e.g. $\mu^2 = 2, \lambda^2 = 1$: $\mathbf{n} \lesssim 0.67$

Thank you!

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Gaussian optimality for energy-constrained entropy rates

Quantum Attenuator: $\mathcal{L}_-(\rho) = a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}$

Theorem

$$\inf_{\rho: \text{Tr}(\hat{n}\rho) \leq \mathbf{n}} \left. \frac{d}{dt} \right|_{t=0} J_-(\rho) = J_-(\omega_{\mathbf{n}}) = -\mathbf{n} \log \left(1 + \frac{1}{\mathbf{n}}\right)$$

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Step 1 (Correspondence to classical problem)

For $\rho = \sum_n p_n |n\rangle\langle n|$ we have

Let $p = (p_0, p_1, \dots)$ be a prob. distribution

$$\rho(t) = e^{t\mathcal{L}_-}(\rho) = \sum_n p_n(t) |n\rangle\langle n|$$

$$\text{with } \dot{p}_n(t) = -np_n(t) + (n+1)p_{n+1}(t)$$

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Denote $(\mathcal{C}_-(p))_n = -np_n + (n+1)p_{n+1}$

$$\text{with } \dot{p}_n(t) = -np_n(t) + (n+1)p_{n+1}(t)$$

Then $p(t) = e^{t\mathcal{C}_-}(p)$ - pure-death process

Theorem

$$\inf_{\rho: \text{Tr}(\hat{n}\rho) \leq \mathbf{n}} \left. \frac{d}{dt} \right|_{t=0} S(e^{t\mathcal{L}_-}(\rho)) = \inf_{p: \mathbb{E}_p[N] \leq \mathbf{n}} \left. \frac{d}{dt} \right|_{t=0} H(e^{t\mathcal{C}_-}(p))$$

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$\longrightarrow H(p) := -\sum_{k=0}^{\infty} p_k \log p_k$

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Step 2 (Classical problem)

$$\inf_{p: \mathbb{E}_p[N] \leq \mathbf{n}} \left. \frac{d}{dt} \right|_{t=0} H(e^{t\mathcal{C}_-}(p)) = -\mathbf{n} \log \left(1 + \frac{1}{\mathbf{n}}\right)$$

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Step 3 (Gaussian optimality)

Let $\omega_{\mathbf{n}}$ be a Gaussian thermal state with a mean-photon number \mathbf{n}

Then

$$\left. \frac{d}{dt} \right|_{t=0} S(e^{t\mathcal{L}_-}(\omega_{\mathbf{n}})) = -\mathbf{n} \log \left(1 + \frac{1}{\mathbf{n}}\right)$$