Markovian Marginals

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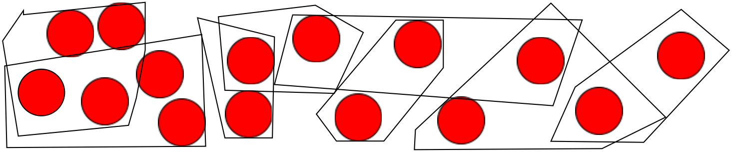
Marginal Problem

Consider a physical system $\Lambda \supset I$.



Given $\{\rho^I \geq 0\}$, is there a $\sigma \geq 0$ such that

$$\sigma^I = \rho^I \quad \forall I?$$

If yes, the marginals are **consistent**.

Marginal Problem: Why do people care?

Suppose $H = \sum_I h_I$.

$$\begin{aligned} E_{gs} &= \min_{\rho} \mathrm{Tr}(\rho H) \\ &= \min_{\rho} \sum_I \mathrm{Tr}(\rho h_I) \\ &= \min_{\text{Consistent } \{\rho^I\}} \sum_I \mathrm{Tr}(\rho^I h_I) \end{aligned}$$

* $\rho, \rho^I \geq 0$. $\mathrm{Tr}(\rho) = \mathrm{Tr}(\rho^I) = 1$.

Marginal Problem: No free lunch

- N-representability problem: QMA-hard
  - Liu, Christandl, Verstraete (2007)
- Consistency problem: QMA-hard
  - Liu (2007)
- What if ρ^I are classical probability distributions?: Still NP-hard
- A respectable senior physicist: People work on the marginal problem for about 10 years, give up on it, and then the next generation repeats the cycle 10 years later

Marginal Problem: where our work stands

- Nonoverlapping marginal problem: restricts the support of the marginals.
  - Bravyi(2003), Klyachko(2004), Hayden and Daftuar(2005), Christandl and Mitchison(2006),$\cdots$
- Sometimes one can show the lack of solution.
  - Osborne(2008), Kim(2012),$\cdots$
- Sometimes the overlapping marginal problem **does** admit a solution:
  - Fannes, Nachtergaele, and Werner(1992), Cramer et al.(2011)
    - Given the marginals of a "reasonable" finitely correlated state/matrix product state, one can efficiently certify their consistency.
  - Markovian marginals

Markovian marginals

At the minimal level of description, Markovian marginal consists of marginals that obey two types of constraints.

- Local consistency: $\mathrm{Tr}_{A\setminus B}(\rho^A) = \mathrm{Tr}_{B\setminus A}(\rho^B)$.
  - Demanded everywhere. Otherwise they cannot be consistent.
- Local Markov: Marginals have an internal quantum Markov chain structure.
  - Needs to be specified further. This is what makes the solution work.

Markovian marginals: Pros and Cons

- Pros
  - The local Markov condition is physically motivated and in fact reasonable.
    - "Physical" states with finite correlation length.
  - More solutions possible(probably)
- Cons
  - No theoretical guarantee on efficient algorithm.
  - Need to be improved to be practical.
    - Ask me later!

Goal of this talk

- Specify a Markovian marginal which is guaranteed to be consistent.
- Explain why the condition is reasonable.
- The main idea behind the proof.

Quantum Markov Chain

- Apologia: There is a beautiful theory of quantum Markov processes initiated by L. Accardi, and pursued by various authors. Unfortunately I was unable to use this (more general) formulation.

For this talk, we say that a tripartite state ρ^{ABC} is a quantum Markov chain if its conditional quantum mutual information $I(A:C|B)_\rho$ is 0.

$$I(A:C|B) := S(\rho^{AB}) + S(\rho^{BC}) - S(\rho^{B}) - S(\rho^{ABC}),$$

where

$$S(\rho) := -\mathrm{Tr}(\rho \log \rho).$$

Quantum Markov Chain

- $I(A : C|B) \geq 0$ by the strong subadditivity of entropy: Lieb and Ruskai(1972)
- $I(A : C|B) = 0$ implies a nontrivial structure: Petz(1983)
  - An exciting recent progress! (Wilde's talk yesterday)
  - More on this later...

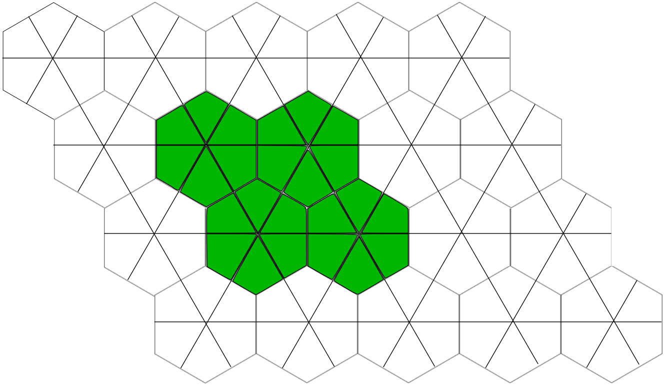
Local Markov chain condition

For a marginal ρ^A, its local Markov condition is formulated as

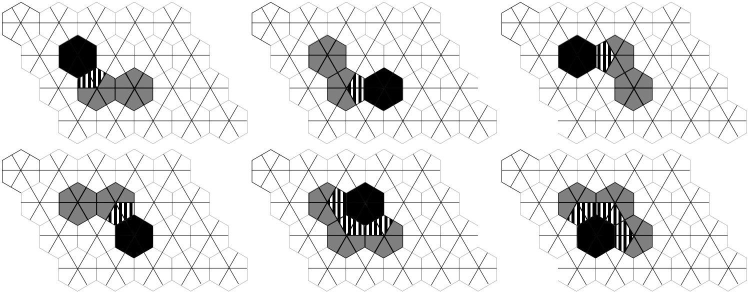
$$I(A_1 : A_2 | A_3)_\rho = 0,$$

where $A = A_1 \cup A_2 \cup A_3$.

Marginals



Local Markov Conditions



Number of conditions

For a translationally invariant system, there are

- 2 local consistency conditions
- 6 local Markov conditions

$$2 + 6 < \infty$$

Within the space of Markovian marginals, energy minimization is a constrained optimization problem with 8 constraints, 2 of which are affine and 6 of which are nonlinear. (Also, don't forget the positive semidefinite constraint!)

Are the conditions reasonable?

According to Kitaev and Preskill, and Levin and Wen's physical argument, 2D systems with a mass gap should obey the following entanglement entropy scaling law:

$$S(\rho^A) = \alpha l - \gamma + \cdots .$$

- The argument is not rigorous. In fact, there are counterexamples.
  - Bravyi(2010?), Zou and Haah(2016)
- But at the same time, it seems to hold in many systems.
- If this is true, the local Markov condition follows.

Comment on the proof

A rough sketch:

1. The local Markov condition implies that the marginals obey a nontrivial set of identities.
2. These identities establish a set of equivalence relations on a certain family of quantum states.
3. Use these equivalene relations.

The difficult part:

- Identifying the right combinatorial object.
  - It is neither the marginal, nor any CP map.
  - The right object is a **collection** of CP maps.
- The combinatorial problem is not a word problem for groups.
  - Partial binary operation, generally no inverse.
  - Even after reducing the problem to a combinatorial problem, you basically need to barrel through this problem brute-force.

Quantum Markov chain admits localized recovery

According to Petz(1983), for ρ^{ABC} with $I(A:C|B)=0$, $\exists$ a CPTP $\Phi : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_{BC})$ which only depends on ρ^{BC} such that

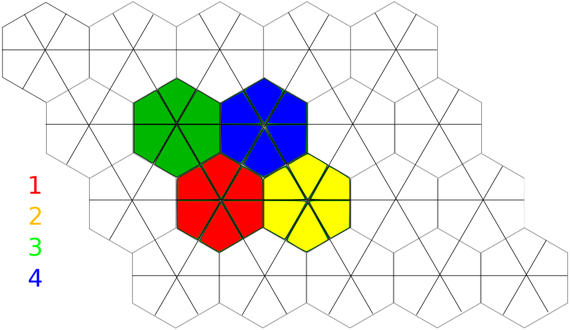
$$\rho^{ABC} = (I_A \otimes \Phi)\rho^{AB}.$$

The recovery map Φ is localized. It acts trivially on A. Moreover, we know that this implication is stable.

- Fawzi, Renner, Sutter, Wilde, Berta, Lemm, Junge, Winter, ...
- Φ is called as the universal recovery map from B to BC.

Local Markov implies nontrivial relations

Local Markov condition endows a nontrivial structure to each marginals, e.g.,



$$1 \to 12 \to 123 \to 1234 = 4 \to 34 \to 234 \to 1234.$$

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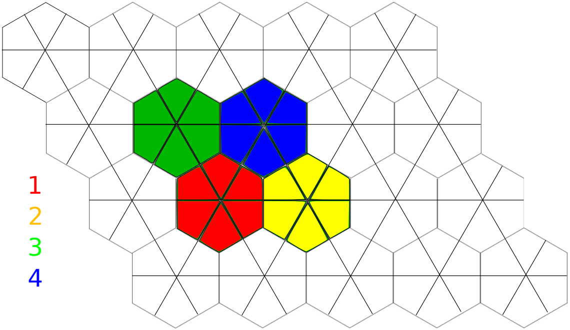


Partial Trace $4[1 \to 12 \to 123 \to 1234] = 1 \to 12 \to 123.$

Certain CP maps “commute”

- For taking a partial trace over two subsystems, their ordering does not matter.
- For applying universal recovery maps supported on disjoint subsystems, their ordering does not matter.
- Similar logic applies between partial trace and universal recovery maps.

* These CP maps technically do not commute, because their compositions are not always well-defined. One needs to carefully adjust the definition of the map.

Relations

A string of elementary cells define a state.

1. For each cell, a collection of universal recovery maps is defined.
2. When a new cell is called, it looks at the existing density matrix, look at its support, and apply the appropriate universal recovery map.
3. The process repeats until the last cell is called.

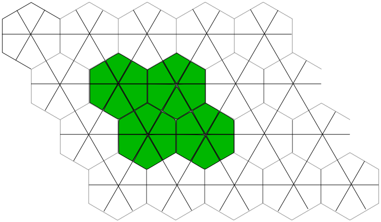
Different strings can give rise to the same state. The equivalence relation is generated by:

- Manifest relations: follows from the "commutativity" of the maps.
- Derived relations: follows from the local Markov condition.

Now what?

We reduced the marginal problem to a combinatorial problem. The combinatorial problem is solved in the following order.

1. From relations involving bounded number of elementary cells, relations involving rows of cells is derived.
2. Two-row reduction.
3. Two-column reduction.
4. Use the derived relations to complete the proof.



Discussion

The states that obey the entanglement entropy scaling law can be described by Markovian marginals, but there is more.

- Maximum global entropy admits a local decomposition.
- Long-range correlations can be also computed efficiently.
- More solutions possible(probably).

Future directions

- Same conclusion from a weaker condition?
- Markovian marginals for quantum chemistry?
- Markovian marginals for inference in classical Bayesian methods?