Quantum Approximate Markov Chains and the Locality of Entanglement Spectrum

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Entanglement in Many-Body Quantum States
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Entanglement Entropy: \( S(A) = -\text{tr}(\rho_A \log \rho_A) \)
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For generic quantum states: \[ S(X) \approx \text{vol}(X) \] (Page ‘93)
Entanglement in Many-Body Quantum States

Entanglement Entropy: $S(A) = -\text{tr}(\rho_A \log \rho_A)$

For generic quantum states: $S(X) \approx \text{vol}(X)$ (Page ‘93)

What’s the behavior of EE for interesting states of matter?
Area Law

Entanglement is “localized”, concentrated around the boundary

For every region $X$:  
\[ S(X) = \alpha |\partial X| - \gamma + \ldots \]

e.g. gapped models, 2+1 CFT (from RT formula)
Area Law

For every region $X$: \[ S(X) = \alpha |\partial X| - \gamma + \ldots \]

$\gamma$: Topological EE
(signature topological order)

$\gamma = \log D$, \[ D = \sqrt{\sum_a d_a^2} \]  
$D$: Quantum dimension

Entanglement is “localized”, concentrated around the boundary
Area Law

For every region $X$:

$$S(X) = \alpha|\partial X| - \gamma + \ldots$$

- Topological EE quantifies “non-local entanglement”

(Kitaev ‘12) $\gamma = 0$: state is adiabatically connected to trivial phase

(Kim ‘13) $\log(N) \leq 2\gamma$ \hspace{1cm} $N :=$ number topologically protected states

Entanglement is “localized”, concentrated around the boundary
Area Law

Entanglement is “localized”, concentrated around the boundary

For every region $X$: $S(X) = \alpha |\partial X| - \gamma + \ldots$

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(Kitaev ‘12) $\gamma = 0$: state is adiabatically connected to trivial phase
(Kim ‘13) $\log(N) \leq 2\gamma$ $N :=$ number topologically protected states

- Bulk-boundary correspondence: topological order in the bulk has an effect on the boundary
Area Law

Entanglement is “localized”, concentrated around the boundary

What are the consequences of an area law?
What’s the influence of TEE on the boundary?
What are the consequences of an area law?  
What’s the influence of TEE on the boundary?  
**This talk:**

Area Law

Entanglement is “localized”, concentrated around the boundary

\[ |\psi\rangle_{AA^c} \]

\( C^2 \)

TEE determines locality of
i) Boundary State
ii) Entanglement Spectrum

by strong subadditivity and stronger subadditivity
Quantum Information 1.01: Fidelity

... it’s a measure of distinguishability between two quantum states.

Given two quantum states their fidelity is given by

\[ F(\rho, \sigma) := \text{tr}( (\rho^{1/2} \sigma \rho^{1/2})^{1/2} ) \]

It tells how distinguishable they are by any quantum Measurement

Ex 1: F=1: same state

Ex 2: F=0 : perfectly distinguishable states
Quantum Information 1.01: Relative Entropy

... it’s another measure of distinguishability between two quantum states.

Def: \( S(\rho \| \sigma) := \text{tr}(\rho (\log(\rho) - \log(\sigma))) \)

Gives optimal exponent for distinguishing the two states.

Pinsker’s inequality: \( S(\rho \| \sigma) \geq -\frac{1}{2} \log F(\rho, \sigma) \)

\( S(\rho \| \sigma) \approx 0 \implies \rho \approx \sigma \)
Topological EE and Locality of Boundary States

\( \rho_{XYZ} \): reduced state on XYZ

XYZ Boundary of A
Topological EE and Locality of Boundary States

$\rho_{XYZ}$: reduced state on XYZ

XYZ Boundary of A

Result 1. If $S(X) = \alpha |\partial X| - \gamma + ...$:

$$\gamma \approx \min_{H_{XY}, H_{YZ}} S(\rho_{XYZ} \| \exp(H_{XY} + H_{YZ})/\text{tr}(...))$$
Topological EE and Locality of Boundary States

\( \rho_{XYZ} \): reduced state on XYZ

XYZ Boundary of A

\( e^{-|\partial X|/\xi} \)

**Result 1.** If \( S(X) = \alpha|\partial X| - \gamma + ... \):

\[
\gamma \approx \min_{H_{XY}, H_{YZ}} S(\rho_{XYZ} || \exp(H_{XY} + H_{YZ})/\text{tr}(...))
\]

\( e^{-|\partial X|/\xi'} \)
Topological EE and Locality of Boundary States

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\[
\gamma \approx \min_{H_{XY}, H_{YZ}} S(\rho_{XYZ\parallel \exp(H_{XY} + H_{YZ})/\text{tr}(...))}
\]

\[
\approx \min_{H_{B_1B_2}, ..., H_{B_{2k-1}B_{2k}}} S(\rho_{B_1...B_{2k}\parallel \exp(H_{B_1B_2} + ... + H_{B_{2k-1}B_{2k}})/\text{tr}(...))}
\]
Topological EE and Locality of Boundary States

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$$\gamma \approx \min_{H_{XY}, H_{YZ}} S(\rho_{XYZ} \| \exp(H_{XY} + H_{YZ})/\text{tr}(\cdot))$$

$$\approx \min_{H_{B_1B_2}, \ldots, H_{B_{2k-1}B_{2k}}} S(\rho_{B_1\ldots B_{2k}} \| \exp(H_{B_1B_2} + \ldots + H_{B_{2k-1}B_{2k}})/\text{tr}(\cdot))$$

$$e^{-|\partial X|/\xi}$$

$l = O(\log(|A|))$
Topological EE and Locality of Boundary States

$\rho_{XYZ}$: reduced state on XYZ

XYZ Boundary of A

Obs 1: $\gamma = 0$

$\implies \rho_B \approx \exp\left(\sum H_{B_i B_j} / \text{tr}(\ldots)\right)$

Obs 2: Thermal states has same on-site symmetries as original state

Obs 3: Thermal state is max entropy state consistent with local constraints
TEE gives number of non-local bits

Interpretation relative entropy (Anshu et al ‘14)

Alice knows $\rho$

Bob knows $\sigma$
TEE gives number of non-local bits

Interpretation relative entropy (Anshu et al. ‘14)

Alice

knows $\rho$

EPR pairs

Bob

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Classical Comm.

Bob

knows $\sigma$
TEE gives number of non-local bits

Interpretation relative entropy \((\text{Anshu et al '14})\)

Alice knows \(\rho\)

EPR pairs

Classical Comm.

Bob knows \(\sigma\)

What’s the minimum classical comm. required for Bob to learn \(\rho\)? (i.e. to be able to prepare a copy of \(\rho\))
TEE gives number of non-local bits

Interpretation relative entropy (Anshu et al. ‘14)

Alice

\[ \approx S(\rho \| \sigma) \]

knows \( \rho \)

EPR pairs

Classical Comm.

Bob

knows \( \sigma \)

necessary and sufficient for Bob to prepare a copy of \( \rho \)
TEE gives number of non-local bits

Interpretation relative entropy (Anshu et al ‘14)

Alice knows $\rho$

$\gamma \approx S(\rho\|\sigma)$ necessary and sufficient for Bob to prepare a copy of $\rho$

Bob knows $\sigma$

$$\gamma \approx \min_{\sigma \in \text{Local Gibbs State}} S(\rho_{B_1\ldots B_{2k}}\|\sigma)$$ gives number of non-local bits of $\rho$

obs: Consistent with $\gamma = \log(\text{quantum dimension})$
Entanglement Spectrum

\[ |\psi\rangle_{AA^c} \]

\[ \mathbb{C}^2 \]

\[ \lambda(\rho_A) : \text{eigenvalues of } \rho_A \]

Entanglement Spectrum

Area law statement about \(- \sum_i \lambda_i \log \lambda_i\)

What can we say about the whole spectrum?
Area law statement about $- \sum_i \lambda_i \log \lambda_i$

What can we say about the whole spectrum?
(Haldane, Li ’08, Cirac, Poiblanc, Schuch, Verstraete ’11, ...)

$\gamma=0$: matches spectrum thermal state local model

$\gamma \neq 0$: matches spectrum thermal state local model after projecting into topological superselection sector
Entanglement Spectrum

We assume translation invariance s.t. $\rho_X = \rho_{X'}$

**Result 2:** If $S(X) = \alpha |\partial X| - \gamma + ...$

\[
\gamma = 0 \implies \lambda(\rho_X)^{\otimes 2} \approx \lambda\left( e \sum_k H_{B_k, B_{k+1}} \right)
\]

\[
\gamma \neq 0 \implies \lambda(\rho_X)^{\otimes 2} \approx \lambda(\sigma),
\]

\[
\text{tr}_{B_1}(\sigma) = e \sum_{k>1} H_{B_k, B_{k+1}}
\]
Result 2 from 1

From area law assumption: (more later)

\[ \rho_{XX'} \approx \rho_X \otimes \rho_{X'} \]
Result 2 from 1

From area law assumption: (more later)

\[ \rho_{XX'} \approx \rho_X \otimes \rho_{X'} \]

\[ \lambda(\rho_{XX'}) = \lambda(\rho_B) \quad \Rightarrow \quad \lambda(\rho_X) \otimes \lambda(\rho_{X'}) \approx \lambda(\rho_B) \]

**Uhlmann’s theorem** There is an isometry \( U : B \rightarrow B_XB_{X'} \) s.t.

\[ U|\psi\rangle_{XB_X} \approx |\phi\rangle_{XB_X} \otimes |\phi'\rangle_{XB_{X'}} \quad \rho_X = \text{tr}_{B_X} (|\phi\rangle\langle\phi|_{XB_X}) \]

\( U \) maps degrees of freedom of \( X \) and \( X' \) into \( B \)
Result 2 from 1

From area law assumption:
(more later)

\[
\rho_{XX'} \approx \rho_X \otimes \rho_{X'}
\]

\[
\lambda(\rho_{XX'}) = \lambda(\rho_B) \rightarrow \lambda(\rho_X) \otimes \lambda(\rho_{X'}) \approx \lambda(\rho_B)
\]

If \( \gamma = 0 \), \( \rho_B \approx e^{\sum_k H_{B_k, B_{k+1}}}/Z \)

\[
\gamma \approx \min_{H_{XY}, H_{YZ}} S(\rho_{XYZ} || \exp(H_{XY} + H_{YZ})/\text{tr}(...))
\]

\[
\approx \min_{H_{B_1B_2}, \ldots, H_{B_{2k-1}B_{2k}}} S(\rho_{B_1\ldots B_{2k}} || \exp(H_{B_1B_2} + \ldots + H_{B_{2k-1}B_{2k}})/\text{tr}(...))
\]
Why does it hold?

We want to show:

\[
\gamma \approx \min_{H_{XY}, H_{YZ}} S(\rho_{XYZ} \| \exp(H_{XY} + H_{YZ})/\text{tr}(...))
\]

\[
\approx \min_{H_{B_1B_2, \ldots, H_{B_{2k-1}B_{2k}}} } S(\rho_{B_1B_2, \ldots, B_{2k-1}B_{2k}} \| \exp(H_{B_1B_2} + \ldots + H_{B_{2k-1}B_{2k}})/\text{tr}(...))
\]

\(\chi = O : \) follow from strong subadditivity (SSA) (Lieb, Ruskai ‘73)

\[
S(AB) + S(BC) \geq S(ABC) + S(B)
\]

\(\chi \neq O : \) follows from a strengthening of SSA (Fawzi and Renner ‘14)
Applications of SSA

Used to prove optimal rates for nearly every quantum information protocol.

- Channel capacities (classical, quantum, private)
- Distillable Entanglement
- ....

(Casini, Huerta, Myers ...) SSA + Lorentz Invariance:
- Entropic proof of the c-theorem
  (irreversibility of renormalization flow)
- Proof of Bekenstein’s and Bousso’s bound

(Ryu-Takayanagi, Headrick, ...) Test for holographic proposals of entropy

Many others...
Conditional Mutual Information

Given $\rho_{ABC}$,

$$I(A : C|B) := S(AB) + S(BC) - S(ABC) - S(B)$$

$$= S(\rho_{ABC} \parallel \exp(\log(\rho_{AB}) + \log(\rho_{BC}) - \log(\rho_B)))$$

Strong subadditivity: $I(A : C|B) \geq 0$
Conditional Mutual Information

Given $\rho_{ABC}$,

$$I(A : C|B) := S(AB) + S(BC) - S(ABC) - S(B)$$

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**Strong subadditivity:**

$$I(A : C|B) \geq 0$$

**Stronger subadditivity** (Fawzi-Renner ’14):

$$I(A : C|B) \geq \frac{1}{2} \min_{\Lambda:B\rightarrow BC} - \log(F(\rho_{ABC}, \Lambda(\rho_{AB})))$$
Conditional Mutual Information

Given $\rho_{ABC}$,

\[
I(A : C|B) := S(AB) + S(BC) - S(ABC) - S(B)
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= S(\rho_{ABC} || \exp(\log(\rho_{AB}) + \log(\rho_{BC}) - \log(\rho_B)))
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Stronger subadditivity (Fawzi-Renner ’14):

\[
I(A : C|B) \geq \frac{1}{2} \min_{\Lambda : B \rightarrow BC} - \log(F(\rho_{ABC}, \Lambda(\rho_{AB})))
\]

\[
I(A : C|B) \approx 0 \implies I_A \otimes \Lambda^{B \rightarrow BC}(\rho_{BC}) \approx \rho_{ABC}
\]

quantum channel
Conditional Mutual Information

Given $\rho_{ABC}$,

$$I(A : C|B) := S(AB) + S(BC) - S(ABC) - S(B) = S(\rho_{ABC} || \exp(\log(\rho_{AB}) + \log(\rho_{BC}) - \log(\rho_{B})))$$

**Strong subadditivity:** $I(A : C|B) \geq 0$

**Stronger subadditivity** (Fawzi-Renner ’14):

$$I(A : C|B) \geq \frac{1}{2} \min_{\Lambda : B \rightarrow BC} - \log(F(\rho_{ABC}, \Lambda(\rho_{AB})))$$

Can reconstruct the state ABC from reduction on AB by acting on B only.

\[ A \quad \overset{\text{B}}{\longrightarrow} \quad A \quad \overset{\text{B}}{\longrightarrow} \quad C \]
Consequence of Area Law: State Reconstruction

For every ABC with trivial topology:

\[ I(A : C | B) \approx 0 \]

\[
I(A : C | B) \\
= S(AB) + S(BC) - S(ABC) - S(B) \\
= \alpha(|\partial(AB)| + |\partial(BC)||\partial(ABC)| - |\partial(B)|) + ... \\
= \alpha(6l + 6l - 8l - 4l) + ...
\]
TEE as Conditional Mutual Info

(Kitaev, Preskill ‘05, Levin, Wen ‘05)

\[
\gamma = I(A : C|B) + \ldots
\]

\[
I(A : C|B) = S(AB) + S(BC') - S(ABC) - S(B)
= \alpha(\partial(AB) + |\partial(BC')| - |\partial(ABC')| - |\partial(B)|) - \gamma - \gamma + \gamma + 2\gamma + \ldots
= \gamma + \ldots
\]

Non zero TEE gives an obstruction to reconstruct \( \rho_{ABC} \) from \( \rho_{AB} \) by acting on B.
Why does it work?

We want to show:

$$\gamma \approx \min_{H_{XY}, H_{YZ}} S(\rho_{XYZ} \| \exp(H_{XY} + H_{YZ})/\text{tr}(\ldots))$$

$$\approx \min_{H_{B_1B_2}, \ldots, H_{B_{2k-1}B_{2k}}} S(\rho_{B_1\ldots B_{2k}} \| \exp(H_{B_1B_2} + \ldots + H_{B_{2k-1}B_{2k}})/\text{tr}(\ldots))$$
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\]

Let’s start with the case \(\gamma = 0\).

Need to show \(\rho_{B_1\ldots B_{2k}}\) is close to thermal assuming all conditional mutual information are small, i.e. approximately independence

\[
I(B_1 \ldots B_{j-1} : B_{j+1} \ldots B_{2k-1} | B_j B_{2k}) \approx 0
\]
X, Y, Z with distribution p(x, y, z)
i) X-Y-Z Markov if X and Z are independent conditioned on Y
ii) X-Y-Z Markov if there is a channel Λ : Y -> YZ s.t. Λ(p_{xy}) = p_{xyz}

iii) \( I(X : Y | Z)_p = E_{z \sim p(z)} I(X : Y)_{p(x, y | z=z')} \)
We say $X_1, \ldots, X_n$ on a graph $G$ form a Markov Network if $X_i$ is independent of all other $X'$s conditioned on its neighbors.

Ex: Markov chains
Hammersley-Clifford Theorem

Markov networks
Hamiltonian

Gibbs state local classical
(on cliques of the graph)
Going Back

Need to show $\rho_{B_1 \ldots B_{2k}}$ is close to thermal assuming all conditional mutual information are small (approximately independence)

$$I(B_1 \ldots B_{j-1} : B_{j+1} \ldots B_{2k-1} | B_j B_{2k}) \approx 0$$

We want a quantum and approximate version of Hammersley-Clifford, but only for 1D chains
Quantum Markov Chain

Classical:  $X, Y, Z$ with distribution $p(x, y, z)$

i)  $X-Y-Z$ Markov if $X$ and $Z$ are independent conditioned on $Y$

ii) $X-Y-Z$ Markov if there is a channel $\Lambda : Y \rightarrow YZ$ s.t. $\Lambda(p_{XY}) = p_{XYZ}$

Quantum:  

i)  $\rho_{ABC}$ Markov quantum state if $A$ and $C$ are ”independent conditioned” on $B$  

(Hayden, Jozsa, Petz, Winter ’03)
Quantum Markov Chain

Classical:  $X, Y, Z$ with distribution $p(x, y, z)$

i)  $X-Y-Z$ Markov if $X$ and $Z$ are independent conditioned on $Y$

ii) $X-Y-Z$ Markov if there is a channel $\Lambda : Y \rightarrow YZ$ s.t. $\Lambda(p_{XY}) = p_{XYZ}$

Quantum:  \cite{Hayden03}

i)  $\rho_{ABC}$ Markov quantum state if $A$ and $C$ are ”independent conditioned” on $B$, i.e. $H_B \simeq \bigoplus_k H_{B_L,k} \otimes H_{B_R,k}$ and

\[
\rho_{ABC} = \bigoplus_k p_k \rho_{AB_L,k} \otimes \rho_{B_R,k} C
\]
Quantum Markov Chain

Classical: $X, Y, Z$ with distribution $p(x, y, z)$

i) $X$-$Y$-$Z$ Markov if $X$ and $Z$ are independent conditioned on $Y$

ii) $X$-$Y$-$Z$ Markov if there is a channel $\Lambda : Y \rightarrow YZ$ s.t. $\Lambda(p_{XY}) = p_{XYZ}$

Quantum: (Hayden, Jozsa, Petz, Winter ’03)

i) $\rho_{ABC}$ Markov quantum state if $A$ and $C$ are ”independent conditioned” on $B$, i.e. $H_B \simeq \bigoplus_k H_{B_L,k} \otimes H_{B_R,k}$ and

$$\rho_{ABC} = \bigoplus_k p_k \rho_{A B_L,k} \otimes \rho_{B_R,k} \rho_C$$

ii) $\rho_{ABC}$ Markov if there is channel $\Lambda : B \rightarrow BC$ s.t. $\Lambda(\rho_{AB}) = \rho_{ABC}$
Quantum Markov Chain

Quantum: (Hayden, Jozsa, Petz, Winter ’03)

i) \( \rho_{ABC} \) Markov quantum state if A and C are “independent conditioned” on B, i.e. \( H_B \simeq \bigoplus_k H_{B_{L,k}} \otimes H_{B_{R,k}} \) and

\[
\rho_{ABC} = \bigoplus_k p_k \rho_{AB_{L,k}} \otimes \rho_{B_{R,k}C}
\]

ii) \( \rho_{ABC} \) Markov if there is channel \( \Lambda : B \to BC \) s.t. \( \Lambda(\rho_{AB}) = \rho_{ABC} \)

iii) \( \rho_{ABC} \) Markov if \( \rho_{ABC} = e^{H_{AB}+H_{BC}}, \quad [H_{AB}, H_{BC}] = 0 \)
Quantum Hammersley-Clifford Theorem

(Leifer, Poulin ‘08, Brown, Poulin ‘12) Analogous result holds replacing classical Hamiltonians by *commuting* quantum Hamiltonians

(obs: quantum version more fragile; only works for graphs with no 3-cliques)

Only Gibbs states of commuting Hamiltonians appear. Is there a fully quantum formulation?
Q. Approximate Markov States

ρ quantum approximate Markov if for every A, B, C

\[ I(A : C \mid B) \rightarrow 0 \text{ when } \text{dist}(A, C) \rightarrow \infty \]

Conjecture
Quantum Approximate Markov \(\leftrightarrow\) Gibbs state local Hamiltonian

\[ \rho = e^{\sum_k H_k} \]
Conjecture
Quantum Approximate Markov Gibbs state local Hamiltonian

(Wolf, Verstraete, Hastings, Cirac ‘07) \[ I(A : BC)_{\rho_T} \leq \frac{c}{T} |\partial A| \]

Gibbs state @ temperature $T$:

\[ \rho_T := e^{-H/T}/Z \]

\[ H = \sum_k H_k, \quad \|H_k\| \leq 1 \]
Strengthening of Area Law

Conjecture
Quantum Approximate Markov \rightarrow Gibbs state local Hamiltonian

From conjecture:

\[ I(A : BC') = I(A : B) + I(A : C|B) \approx I(A : B) \]

Gives rate of saturation of area law
Approximate Quantum Markov Chains are Thermal

**Thm**

1. Let $H$ be a local Hamiltonian on $n$ qubits. Then

$$I(A : C | B)_{\rho_T} \leq e^{-c' \sqrt{|B|}} + e^{c/T}$$
Approximate Quantum Markov Chains are Thermal

thm

1. Let $H$ be a local Hamiltonian on $n$ qubits. Then

$$I(A : C | B)_{\rho_T} \leq e^{-c' \sqrt{|B|}} + e^{c/T}$$

2. Let $\rho_1...n$ be a state on $n$ qubits s.t. for every split ABC with $|B|$, $I(A : C | B) \leq \varepsilon$. Then

$$\min_{H \in \mathcal{H}_{2m}} S(\rho || e^H) \leq \varepsilon \frac{n}{m}$$

$$\mathcal{H}_{2m} := \{ H : H = \sum_k H_{k,k+1}, \forall k \text{ supp}(H_{k,k+1}) \leq 2m \}$$
Proof Part 2

Let $\sigma X_1 \ldots X_{\frac{n}{m}}$ be the maximum entropy state s.t.

$$\sigma X_i, X_{i+1} = \rho X_i, X_{i+1} \quad \forall i \in \left[\frac{n}{m}\right]$$
Proof Part 2

Let $\sigma X_1 \ldots X_n$ be the maximum entropy state s.t.

$$\sigma X_i, X_{i+1} = \rho X_i, X_{i+1} \quad \forall i \in [n/m]$$

Fact 1 (Jaynes ‘57): $\sigma = e^{\sum_k H_{X_k, X_{k+1}}}$

“maximum entropy state given linear constraints is thermal”

$$\argmax (S(\sigma) \text{ s.t. } \text{tr}(\sigma M_i) = c_i) = \exp \left( \sum_i \lambda_i M_i \right)$$
Proof Part 2

Let $\sigma X_1 \ldots X_{\frac{n}{m}}$ be the maximum entropy state s.t.

$$\sigma_{X_i, X_{i+1}} = \rho_{X_i, X_{i+1}} \quad \forall i \in [n/m]$$

Fact 1 (Jaynes ‘57): $\sigma = e \sum_k H_{X_k, X_{k+1}}$

Fact 2 $\min_{H \in \mathcal{H}_{2m}} S(\rho || e^H / Z) \leq -S(\rho) - \text{tr}(\rho \log \sigma)$

$= S(\sigma) - S(\rho)$

Let’s show it’s small
Proof Part 2

\[
S(X_1 \cdots X_{n/m})_{\sigma} \\
\leq S(X_1 X_2)_{\sigma} - S(X_2)_{\sigma} + S(X_2 \cdots X_{n/m})_{\sigma}
\]

SSA
Proof Part 2

\[ S(X_1 \ldots X_{n/m})_\sigma \]
\[ \leq S(X_1 X_2)_\sigma - S(X_2)_\sigma + S(X_2 \ldots X_{n/m})_\sigma \]
\[ \leq S(X_1 X_2)_\sigma - S(X_2)_\sigma + S(X_2 X_3)_\sigma - S(X_3)_\sigma + S(X_3 \ldots X_{n/m})_\sigma \]
Proof Part 2

\[ S(X_1 \ldots X_{n/m})_\sigma \]
\[ \leq S(X_1X_2)_\sigma - S(X_2)_\sigma + S(X_2 \ldots X_{n/m})_\sigma \]
\[ \leq S(X_1X_2)_\sigma - S(X_2)_\sigma + S(X_2X_3)_\sigma - S(X_3)_\sigma + S(X_3 \ldots X_{n/m})_\sigma \]
\[ \leq \sum_i S(X_iX_{i+1})_\sigma - S(X_{i+1})_\sigma \]
Proof Part 2

\[ S(X_1 \ldots X_{n/m})_\sigma \]
\[ \leq S(X_1 X_2)_\sigma - S(X_2)_\sigma + S(X_2 \ldots X_{n/m})_\sigma \]
\[ \leq S(X_1 X_2)_\sigma - S(X_2)_\sigma + S(X_2 X_3)_\sigma - S(X_3)_\sigma + S(X_3 \ldots X_{n/m})_\sigma \]
\[ \leq \sum_{i} S(X_i X_{i+1})_\sigma - S(X_{i+1})_\sigma \]
\[ = \sum_{i} S(X_i X_{i+1})_\rho - S(X_{i+1})_\rho \]

Since \( \sigma_{X_i, X_{i+1}} = \rho_{X_i, X_{i+1}} \quad \forall i \in [n/m] \)
Proof Part 2

\[ S(X_1 \ldots X_{n/m})_\sigma \]
\[ \leq S(X_1X_2)_\sigma - S(X_2)_\sigma + S(X_2 \ldots X_{n/m})_\sigma \]
\[ \leq S(X_1X_2)_\sigma - S(X_2)_\sigma + S(X_2X_3)_\sigma - S(X_3)_\sigma + S(X_3 \ldots X_{n/m})_\sigma \]
\[ \leq \sum_i S(X_iX_{i+1})_\sigma - S(X_{i+1})_\sigma \]
\[ = \sum_i S(X_iX_{i+1})_\rho - S(X_{i+1})_\rho \]
\[ \leq S(X_1 \ldots X_{n/m})_\rho + \varepsilon \frac{n}{m} \]

Since \( I(X_i : X_{i+2} \ldots X_{n/m} | X_{i+1}) \leq \varepsilon \forall i \)
Proof Part 1

Recap: Let $H$ be a local Hamiltonian on $n$ qubits. Then

$$I(A : C|B)_{\rho_T} \leq e^{-c}\sqrt{|B|} + e^{c/T}$$

We show there is a recovery channel from $B$ to $BC$ reconstructing the state on $ABC$ from its reduction on $AB$.

Summary

• Locality of EE (area law) implies locality of boundary states and entanglement spectrum

• Quantum Approximate Markov Chains are Thermal
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Open Questions:

• Applications to high energy/holography?

• Are two copies of entanglement spectrum needed?

• Is the conjecture about approximate Markov chains true?

• Thermal state has same symmetries as original state. Mapping from 2D (zero temperature) to 1D (thermal). Is it useful for classification of (symmetry-protected) phases?
There exists an operator $X_B$ such that

\[ \rho_{AB}H_{ABC} \approx \text{id}_A \otimes \kappa_{BC} (\rho_{AB}H_{ABC}) = X_B (\text{tr}_B [X_B (\rho_{AB}H_{ABC} (X_B^{-1})^\dagger ) \otimes \rho_H B_R] X_B^\dagger) \]
Structure of Recovery Map

There exists an operator $X\downarrow B$ such that

$$\rho^{\uparrow H\downarrow ABC} \approx \text{id}^{\downarrow A} \otimes \kappa^{\downarrow B \rightarrow BC} \left( \rho^{\downarrow \overline{A} \uparrow B^{\uparrow H\downarrow ABC}} \right) = X^{\downarrow B} \left( \text{tr}^{\downarrow B^\uparrow R} \left[ X^{\downarrow B^\uparrow -1^{\uparrow \dagger}} \rho^{\downarrow \overline{A} \uparrow B^{\uparrow H\downarrow ABC}} \left( X^{\downarrow B^\uparrow -1^{\uparrow \dagger}} \right)^{\dagger} \right] \otimes \rho^{\uparrow H\downarrow B^\uparrow R} \right) X^{\downarrow B^{\uparrow \dagger}}$$

**Difficulty:** $\kappa^{\downarrow B \rightarrow BC}$ is a trace-increasing map
Repeat-until-success Method

We normalize $\kappa \downarrow B \rightarrow BC$ and define a CPTD-map $\Lambda \downarrow B \rightarrow BC$. 
→ Succeed to recover with a constant probability $p$.

Choose $N \sim l \ (|B| = O(l^2))$.
→ Total error = Fail probability $(1-p)^l +$ approx. error $O(e^l - O(l)) = O(e^l - O(l))$. 

\[ A \quad B \downarrow \quad B \downarrow \quad B \downarrow \quad B \downarrow \quad B \downarrow \quad B \downarrow \quad B \downarrow \quad B \downarrow \quad B \downarrow \quad B \downarrow \quad B \downarrow \quad C \]

Apply $\Lambda \downarrow B \downarrow 1$ 
→ Success 

Trace out $B \downarrow 1$ 
$B \downarrow 1 \ C$ & apply $\Lambda$ 

$\downarrow B \downarrow 2 \rightarrow B \downarrow 2 \ B$ 
$\uparrow B \downarrow 1 \ C$ 

Success 

Trace out $B \downarrow N$ 
$-1 \ldots C$ & apply $\Lambda$ 

$\downarrow B \downarrow N$ 

Success 

Fail 

Fail 

Fail
Locality of Perturbations

The key point in the proof:

For a short-ranged Hamiltonian $H$, the local perturbation to $H$ only perturb the Gibbs state locally.

A useful lemma by Araki (Araki, ’69)
For 1D Hamiltonian with short-range interaction $H$,

$$
\| e^{\uparrow H + V} e^{\downarrow - H} - e^{\uparrow H \downarrow I + V} e^{\uparrow - H \downarrow I} \| \leq O(e^{\uparrow - O(I)})
$$

$$
 e^{\uparrow - \beta H} \rightarrow e^{\uparrow - \beta (H+V)} \approx X \downarrow I e^{\uparrow - \beta H} X \downarrow I \uparrow
$$

$$
 X \downarrow I = e^{\uparrow - \beta / 2} (H \downarrow I + V) e^{\uparrow \beta / 2} H \downarrow I
$$

← Local
Proof for $\gamma \neq 0$

**thm 1** Suppose $\ket{\psi}$ satisfies the area law assumption. Then

$$2\gamma \approx I(A : C | B)$$

$$\approx \min_{H_{AB}, H_{BC}} S(\rho_{ABC} \| \exp(H_{AB} + H_{BC})/Z)$$
Proof for $\gamma \neq 0$

We follow the strategy of (Kato et al ‘15) for the zero-correlation length case.

Area Law implies

\[ I(A : B_2 | B_1) \approx 0 \]
\[ I(C : B_1 | B_2) \approx 0 \]

By Fawzi-Renner Bound, there are channels

\[ \Lambda : B_1 \rightarrow B_1 A \]
\[ \Delta : B_2 \rightarrow B_2 C \]

\[ \Lambda(\rho_{B_1 B_2}) \approx \rho_{AB_1 B_2}, \quad \Delta(\rho_{B_1 B_2}) \approx \rho_{B_1 B_2 C} \]
Proof for $\gamma \neq 0$

Define: $\sigma_{AB_1B_2C} := \Lambda^{B_1 \rightarrow B_1A} \otimes \Delta^{B_2 \rightarrow B_2C}(\rho_{B_1B_2})$

We have $\rho_{AB} \approx \sigma_{AB}, \rho_{BC} \approx \sigma_{BC}$

It follows that C can be reconstructed from B. Therefore

$I(A : C|B)_\sigma \approx 0$
Define: $\sigma_{AB_1B_2C} := \Lambda_{B_1 \rightarrow B_1A} \otimes \Delta_{B_2 \rightarrow B_2C}(\rho_{B_1B_2})$

We have $\rho_{AB} \approx \sigma_{AB}, \rho_{BC} \approx \sigma_{BC}$

It follows that C can be reconstructed from B. Therefore

$$I(A : C|B)_\sigma \approx 0$$

Since

$$I(A : C|B)_\sigma = S(\sigma_{ABC}||\exp(\log(\sigma_{AB})) + \log(\sigma_{BC})) - \log(\sigma_B))$$

$\pi \approx \sigma$ with

$$\pi := \exp(\log(\sigma_{AB}) + \log(\sigma_{BC}) - \log(\sigma_B))/\text{tr}(\ldots)$$

So $I(A : C|B)_\pi \approx 0$
Proof for $\gamma \neq 0$

Since $I(A : C | B)_{\pi} \approx 0$

$S(ABC)_{\pi} \approx S(AB)_{\pi} + S(BC)_{\pi} - S(B)_{\pi}$

$\approx S(AB)_{\rho} + S(BC)_{\rho} - S(B)_{\rho}$

$= S(ABC)_{\rho} + I(A : C | B)_{\rho}$

Let $R_2$ be the set of Gibbs states of Hamiltonians $H = H_{AB} + H_{BC}$. Then

$\min_{\nu \in R_2} S(\rho || \nu) = \min_{\nu \in R_2} -S(\rho) - \text{tr}(\rho \log \nu)$

$\approx I(A : C | B)_{\rho} + \min_{\nu \in R_2} -S(\pi) - \text{tr}(\rho \log \nu)$

$\approx I(A : C | B)_{\rho} + \min_{\nu \in R_2} -S(\pi) - \text{tr}(\pi \log \nu)$

$= I(A : C | B)_{\rho}$
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