Local Density Approximation for the Almost-bosonic Anyon Gas

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Many-body Systems and Statistical Mechanics

joint work with D. Lundholm (Stockholm) and N. Rougerie (Grenoble)

Outline

1. Introduction:
   - Fractional statistics and anyons;
   - Almost-bosonic limit for extended anyons and the Average Field (AF) functional [LR];
   - Minimization of the AF functional.
2. Main results [CLR]:
   - Existence of the Thermodynamic Limit (TL) for homogeneous anyons;
   - Local density approximation of the AF functional in terms of a Thomas-Fermi (TF) effective energy.

Main References

- [LR] D. Lundholm, N. Rougerie, *J. Stat. Phys.* **161** (2015).
- [CLR] MC, D. Lundholm, N. Rougerie, in preparation.

Anyons

- The wave function $\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ of identical particles must satisfy
$$|\Psi(\ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots)|^2 = |\Psi(\ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots)|^2 \,;$$
- In 3D there are only 2 possible choices:
$$\Psi(\ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots) = \pm \Psi(\ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots);$$
- In 2D there are other options, related to the way the particles are exchanged (braid group).

Fractional Statistics (Anyons)

For any $\alpha \in [-1, 1]$ (statistics parameter), it might be

$$\boxed{\Psi(\ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots) = e^{i\pi\alpha} \Psi(\ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots)}$$

- $\alpha = 0 \Longrightarrow$ bosons and $\alpha = 1 \Longrightarrow$ fermions;
- anyonic quasi-particle are expected to describe effective excitations in the fractional quantum Hall effect [*physics*; Lundholm, Rougerie '16].

Anyons

- One can work on wave functions Ψ satisfying the anyonic condition (anyonic gauge): complicate because Ψ is not in general single-valued.
- Equivalently one can associate to any anyonic wave function Ψ a bosonic (resp. fermionic) one $\tilde{\Psi} \in L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$ via

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \prod_{j<k} e^{i\alpha\phi_{jk}} \tilde{\Psi}(\mathbf{x}_1, \ldots, \mathbf{x}_N), \quad \phi_{jk} = \arg \frac{\mathbf{x}_j - \mathbf{x}_k}{|\mathbf{x}_j - \mathbf{x}_k|}.$$

Magnetic Gauge

On $L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$ the Schrödinger operator $\sum \left(-\Delta_j + V(\mathbf{x}_j)\right)$ is mapped to

$$H_N = \sum_{j=1}^{N} \left[\left(-i\nabla_j + \alpha \mathbf{A}_j\right)^2 + V(\mathbf{x}_j) \right]$$

with Aharonov-Bohm magnetic potentials $\mathbf{A}_j = \mathbf{A}(\mathbf{x}_j) := \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^{\perp}}{|\mathbf{x}_j - \mathbf{x}_k|^2}$.

AF Approximation

- If the number of anyons is larger, i.e., $N \to \infty$ but at the same time $\alpha \sim N^{-1}$, then one expects a mean-field behavior, i.e.,
$$\alpha \mathbf{A}_j \simeq (N\alpha) \int_{\mathbb{R}^2} \mathrm{d}\mathbf{y} \, \frac{(\mathbf{x}-\mathbf{y})^{\perp}}{|\mathbf{x}-\mathbf{y}|^2} \rho(\mathbf{y}),$$
with ρ the one-particle density associated to $\Phi \in L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$;
- We should then expect that
$$\frac{1}{N} \langle \Phi | \, H_N \, | \Phi \rangle \simeq \mathcal{E}^{\mathrm{af}}_{N\alpha}[u],$$
for some $u \in L^2(\mathbb{R}^2)$ such that $|u|^2(\mathbf{x}) = \rho(\mathbf{x})$ (self-consistency).

AF Functional

$$\boxed{\mathcal{E}^{\mathrm{af}}_{\beta}[u] = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \, \left\{ \left| \left(-i\nabla + \beta \mathbf{A}[|u|^2] \right) u \right|^2 + V|u|^2 \right\}}$$

with $\mathbf{A}[\rho] = \nabla^{\perp} \left(w_0 * \rho \right)$ and $w_0(\mathbf{x}) := \log|\mathbf{x}|$.

Minimization of $\mathcal{E}_\beta^{\mathrm{af}}$

$$\mathcal{E}_\beta^{\mathrm{af}}[u] = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \left\{ \left| \left(-i\nabla + \beta \mathbf{A}[|u|^2] \right) u \right|^2 + V|u|^2 \right\}, \qquad \mathbf{A}[\rho] = \nabla^\perp (w_0 * \rho)$$

- Thanks to the symmetry $u, \beta \longrightarrow u^*, -\beta$, we can assume $\beta \geqslant 0$;
- The domain of $\mathcal{E}_\beta^{\mathrm{af}}$ is $\mathscr{D}[\mathcal{E}^{\mathrm{af}}] = H^1(\mathbb{R}^2)$, since by 3-body Hardy inequality

$$\int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \, \left| \mathbf{A}[|u|^2] \right|^2 |u|^2 \leqslant C \left\| u \right\|_{L^2(\mathbb{R}^2)}^4 \left\| \nabla |u| \right\|_{L^2(\mathbb{R}^2)}^2 .$$

Proposition (Minimization [Lundholm, Rougerie '15])

For any $\beta \geqslant 0$, there exists a minimizer $u_\beta^{\mathrm{af}} \in \mathscr{D}[\mathcal{E}^{\mathrm{af}}]$ of the functional $\mathcal{E}_\beta^{\mathrm{af}}$:

$$E_\beta^{\mathrm{af}} := \inf_{\|u\|_2 = 1} \mathcal{E}_\beta^{\mathrm{af}}[u] = \mathcal{E}_\beta^{\mathrm{af}}[u_\beta^{\mathrm{af}}].$$

Almost-bosonic Limit

- Consider $N \to \infty$ non-interacting anyons with statistics parameter $\alpha = \dfrac{\beta}{N-1}$ for some $\beta \in \mathbb{R}$, i.e., in the almost-bosonic limit;
- Assume that the anyons are extended, i.e., the fluxes are smeared over a disc of radius $R = N^{-\gamma}$.

Theorem (AF Approximation [Lundholm, Rougerie '15])

Under the above hypothesis and assuming that V is trapping and $\gamma \leqslant \gamma_0$,

$$\boxed{\lim_{N\to\infty} \frac{\inf \sigma(H_{N,R})}{N} = \inf_{\|u\|_2=1} \mathcal{E}^{\mathrm{af}}_{\beta}[u]}$$

and the one-particle reduced density matrix of any sequence of ground states of $H_{N,R}$ converges to a convex combination of projectors onto AF minimizers.

Motivations

- The AF approximation is used heavily in physics literature, but typically the nonlinearity is resolved by picking a given ρ (usually the constant density);
- As expected, when $\beta \to 0$, the anyonic gas behaves as a Bose gas.
- More interesting is the regime $\beta \to \infty$, i.e., "less-bosonic" anyons:
  - what is the energy asymptotics of E_β^{af}?
  - is $|u_\beta^{\mathrm{af}}|^2$ almost constant in the homogeneous case, i.e., for $V = 0$ and confinement to a bounded region?
  - how does the inhomogeneity of V modify the density $|u_\beta^{\mathrm{af}}|^2$?
  - what is u_β^{af} like? in particular how does its phase behave?
- The AF functional is not the usual mean-field-type energy (e.g., Hartree or Gross-Pitaevskii), since the nonlinearity depends on the density but acts on the phase of u via a magnetic field.

Homogeneous Gas

- Let $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected set with Lipschitz boundary;
- We consider the following two minimization problems

$$E_{\mathrm{N/D}}(\Omega, \beta, M) := \inf_{u \in H_0^1(\Omega), \|u\|_2 = M} \int_\Omega \mathrm{d}\mathbf{x} \, \left| \left(-i\nabla + \beta \mathbf{A}[|u|^2] \right) u \right|^2 .$$

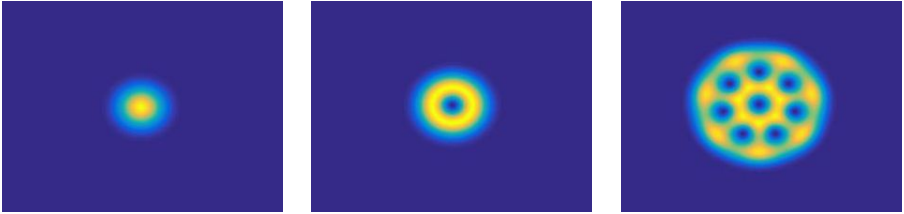
- We want to study the limit $\beta \to \infty$ of $E_{\mathrm{N/D}}(\Omega, \beta, M)/\beta$;
- The above limit is equivalent to the TD limit ($\beta, \rho \in \mathbb{R}^+$ fixed)

$$\lim_{L \to \infty} \frac{E_{\mathrm{N/D}}(L\Omega, \beta, \rho L^2 |\Omega|)}{L^2 |\Omega|} .$$

Lemma (Scaling Laws)

For any $\lambda, \mu \in \mathbb{R}^+$, $E_{\mathrm{N/D}}(\Omega, \beta, M) = \frac{1}{\lambda^2} E_{\mathrm{N/D}} \left(\mu\Omega, \frac{\beta}{\lambda^2 \mu^2}, \lambda^2 \mu^2 M \right)$.

Heuristics ($\beta \gg 1$)



Numerical simulations by R. Duboscq (Toulouse): plot of $|u_\beta^{\mathrm{af}}|^2$ for $\beta = 25, 50, 200$.

- In the homogeneous case, $|u_\beta^{\mathrm{af}}|^2$ can be constant only in a very weak sense (say in L^p, $p < \infty$ not too large);
- The phase of u_β^{af} should contain vortices (with $\# \sim \beta$) almost uniformly distributed with average distance $\sim \frac{1}{\sqrt{\beta}}$ (Abrikosov lattice).

TD Limit

Theorem (∃ TD Limit [MC, Lundholm, Rougerie '16])

Under the above hypothesis on Ω *and for any* $\beta, \rho \in \mathbb{R}^+$, *the limits*

$$e(\beta,\rho) := \lim_{L\to\infty} \frac{E_{\mathrm{N/D}}(L\Omega,\beta,\rho L^2|\Omega|)}{L^2|\Omega|} = \beta|\Omega| \lim_{\tilde{\beta}\to\infty} \frac{E_{\mathrm{N/D}}(\Omega,\tilde{\beta},\rho)}{\tilde{\beta}}$$

exist, coincide and are independent of Ω. *Moreover*

$$e(\beta,\rho) = \beta\rho^2 e(1,1)$$

- $e(1,1)$ is a finite quantity satisfying the lower bound
$$e(1,1) \geqslant 2\pi$$
which follows from the inequality for $u \in H_0^1$
$$\left\| \left(-i\nabla + \beta \mathbf{A}[|u|^2]\right) u \right\|_{L^2(\Omega)}^2 \geqslant 2\pi|\beta| \, \|u\|_{L^4(\Omega)}^4 .$$

Trapped Anyons

$$\mathcal{E}^{\mathrm{af}}_{\beta}[u] = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \, \left\{ \left| \left(-i\nabla + \beta \mathbf{A}[|u|^2] \right) u \right|^2 + V|u|^2 \right\}, \qquad \mathbf{A}[\rho] = \nabla^{\perp} \left(w_0 * \rho \right)$$

- Let $V(\mathbf{x})$ be a smooth homogenous potential of degree $s \geqslant 1$, i.e.,

$$V(\lambda \mathbf{x}) = \lambda^s V(\mathbf{x}), \qquad V \in C^{\infty}(\mathbb{R}^2),$$

  and such that $\min_{|\mathbf{x}| \geqslant R} V(\mathbf{x}) \xrightarrow[R \to \infty]{} +\infty$ (trapping potential).

- We consider the minimization problem for $\beta \gg 1$

$$E^{\mathrm{af}}_{\beta} = \inf_{u \in \mathscr{D}[\mathcal{E}^{\mathrm{af}}], \|u\|_2 = 1} \mathcal{E}^{\mathrm{af}}_{\beta}[u],$$

  with $\mathscr{D}[\mathcal{E}^{\mathrm{af}}] = H^1(\mathbb{R}^2) \cap \left\{ V|u|^2 \in L^1(\mathbb{R}^2) \right\}$ and u^{af}_{β} any minimizer.

- Since $B(\mathbf{x}) = \beta \mathrm{curl} \mathbf{A}[\rho] = 2\pi\beta\rho(\mathbf{x})$, if one could minimize the magnetic energy alone, the effective functional for $\beta \gg 1$ should be

$$\int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \left[B(\mathbf{x}) + V(\mathbf{x}) \right] \rho = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \left[2\pi\beta\rho^2 + V(\mathbf{x})\rho \right].$$

TF Approximation

TF Functional

The limiting functional for $\mathcal{E}_{\beta}^{\mathrm{af}}$ is

$$\mathcal{E}_{\beta}^{\mathrm{TF}}[\rho] := \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \left[e(1,1)\beta\rho^2(\mathbf{x}) + V(\mathbf{x})\rho(\mathbf{x}) \right]$$

with ground state energy $E_{\beta}^{\mathrm{TF}} := \inf_{\|\rho\|_1=1} \mathcal{E}_{\beta}^{\mathrm{TF}}[\rho]$ and minimizer $\rho_{\beta}^{\mathrm{TF}}(\mathbf{x})$.

- Under the hypothesis we made on V, we have
$$E_{\beta}^{\mathrm{TF}} = \beta^{\frac{s}{s+2}} E_{1}^{\mathrm{TF}}, \qquad \rho_{\beta}^{\mathrm{TF}}(\mathbf{x}) = \beta^{-\frac{2}{s+2}} \rho_{1}^{\mathrm{TF}} \left(\beta^{-\frac{1}{s+2}} \mathbf{x} \right).$$
- Given the chemical potential $\mu_{1}^{\mathrm{TF}} := E_{1}^{\mathrm{TF}} + e(1,1) \left\| \rho_{1}^{\mathrm{TF}} \right\|_2^2$, we have
$$\rho_{1}^{\mathrm{TF}}(\mathbf{x}) = \frac{1}{2e(1,1)} \left[\mu_{1}^{\mathrm{TF}} - V(\mathbf{x}) \right]_{+}.$$

Local Density Approximation

Theorem (TF Approx. [MC, Lundholm, Rougerie '16])

Under the hypothesis on V,

$$\lim_{\beta\to\infty} \frac{E^{\mathrm{af}}_{\beta}}{\beta^{\frac{s}{s+2}} E^{\mathrm{TF}}_{1}} = 1, \qquad \beta^{\frac{2}{s+2}} \left| u^{\mathrm{af}}_{\beta} \right|^2 \left(\beta^{\frac{1}{s+2}} \mathbf{x} \right) \xrightarrow[\beta\to\infty]{\mathscr{W}_1} \rho^{\mathrm{TF}}_{1}(\mathbf{x})$$

in the space $\mathscr{W}_1$ of probability measure on $\mathbb{R}^2$ with the Wasserstein distance.

- The result applies to more general potentials, e.g., asymptotically homogeneous potentials;
- The homogeneous case (confinement to Ω, $V = 0$) is included: we recover the asymptotics $E_{\mathrm{N/D}}(\Omega, \beta, 1)/\beta \longrightarrow e(1,1)/|\Omega|$ and

$$\left| u^{\mathrm{af}}_{\beta} \right|^2 (\mathbf{x}) \xrightarrow[\beta\to\infty]{\mathscr{W}_1} \rho^{\mathrm{TF}}_{1}(\mathbf{x}) \equiv |\Omega|^{-1/2}.$$

Local Density Approximation

Theorem (LDA [MC, Lundholm, Rougerie '16])

Under the hypothesis on V, *for any* $\mathbf{x}_0 \in \mathbb{R}^2$ *and any* $0 \leqslant \eta < \frac{s+1}{2(s+2)}$,

$$\sup_{\phi \in C_0(\mathbb{R}^2),\, L(\phi) \leqslant 1} \left| \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\, \phi\left(\beta^{\eta}(\mathbf{x} - \mathbf{x}_0)\right) \left[\rho^{\mathrm{af}}_{\beta}(\mathbf{x}) - \rho^{\mathrm{TF}}_{1}(\mathbf{x}) \right] \right| \xrightarrow[\beta \to \infty]{} 0$$

where $\rho^{\mathrm{af}}_{\beta}(\mathbf{x}) := \beta^{\frac{2}{s+2}} \left| u^{\mathrm{af}}_{\beta} \right|^2 \left(\beta^{\frac{1}{s+2}} \mathbf{x} \right)$.

- $\rho^{\mathrm{af}}_{\beta}$ is well approximated in weak sense by ρ^{TF}_{1} on any scale up to $\beta^{-\eta}$ (in the homogeneous case $1/\sqrt{\beta}$);
- One can not presumably go beyond that scale because that is the mean distance between vortices.

- AF functional:
  - Obtain more information about $e(1,1)$;
  - Investigate the vortex structure of $u_\beta^{\rm af}$, which should be given by some Abrikosov lattice (which in turn is expected to provide info on $e(1,1)$).
- Anyon gas:
  - Recover the behavior $\beta \to \infty$ at the many-body level, in a scaling limit $N \to \infty$, $\alpha = \alpha(N)$ and find out the parameter region where it emerges;
  - Prove the existence of the thermodynamic limit in the same setting.

Thank you for the attention!

- The operator H_N is too singular to be defined on $L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$, but one can remove the planes $\mathbf{x}_j = \mathbf{x}_k$;
- Equivalently one can consider extended anyons, i.e., smear the Aharonov-Bohm fluxes over disc of radius R:

$$\mathbf{A}_j \longrightarrow \mathbf{A}_{j,R} = \sum_{k \neq j} \left(\nabla^{\perp} w_R \right) (\mathbf{x}_j - \mathbf{x}_k),$$

with $w_R(\mathbf{x}) := \frac{1}{\pi R^2} \left(\log |\cdot| * \mathbb{1}_{\mathcal{B}_R(0)} \right) (\mathbf{x})$.

Extended Anyons

For any $R > 0$ the operator

$$\boxed{H_N = \sum_{j=1}^{N} \left[\left(-i \nabla_j + \alpha \mathbf{A}_{j,R} \right)^2 + V(\mathbf{x}_j) \right]}$$

with magnetic potentials $\mathbf{A}_{j,R}$ is self-adjoint on $\mathscr{D}(H_N) \cap L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$.

Proof (Homogeneous Gas)

① $\exists$ of TD limit when Ω is a unit square with Dirichlet b.c.;
② $E_{\mathrm{D}}(LQ, \beta, \rho L^2) - E_{\mathrm{N}}(LQ, \beta, \rho L^2) = o(L^2)$ for squares (IMS);
③ Prove $\exists$ of TD for general domains Ω by localizing into squares.

- Key observation for ① & ③: the magnetic field generated by a bounded region can be gauged away outside (Newton's theorem);
- Pick a smooth and radial f with $\mathrm{supp}(f) \subset \mathcal{B}_\delta(0)$ and N points so that $|\mathbf{x}_j - \mathbf{x}_k| > 2\delta$: consider then the trial state

$$u(\mathbf{x}) = \sum_{j=1}^{N} f(\mathbf{x} - \mathbf{x}_j) e^{-i\phi_j}, \qquad \|u\|_2^2 = N \|f\|_2^2;$$

- In $\{|\mathbf{x} - \mathbf{x}_j| \leqslant \delta\}$ the magnetic field generated by the other discs is

$$\sum_{k \neq j} \nabla^{\perp} \left(w_0 * |f(\mathbf{x} - \mathbf{x}_k)|^2 \right) = \|f\|_2^2 \, \nabla \sum_{k \neq j} \arg(\mathbf{x} - \mathbf{x}_k) =: \nabla \phi_j.$$