Uni-directional quantum graphs

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Joint work with **M. Akila**

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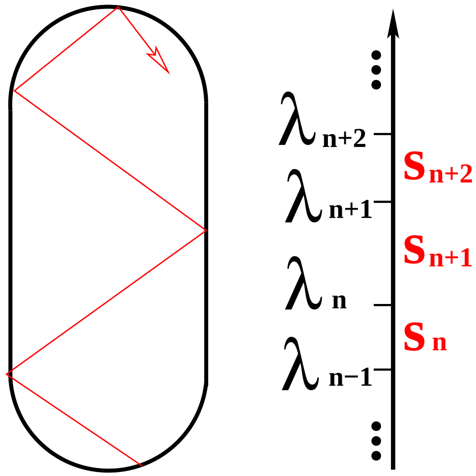
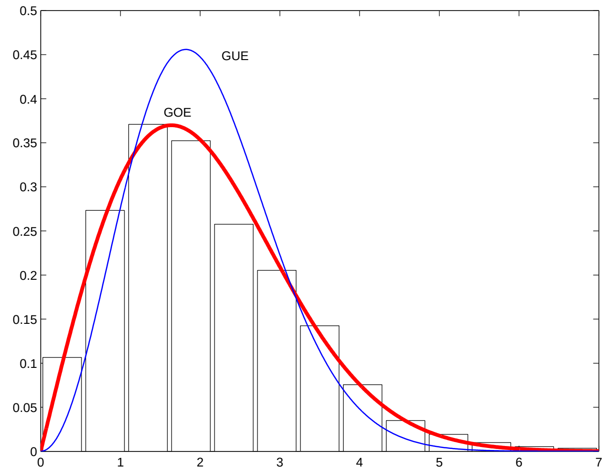
Spectral Universality





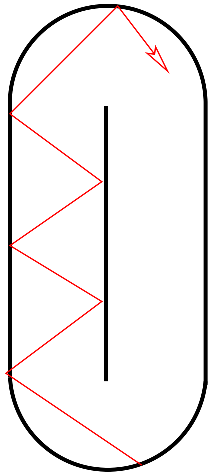
$$(\Delta + \lambda_n)\,\varphi_n = 0, \qquad \varphi_n|_{\partial\Omega} = 0, \quad \varphi_n \in L^2(\Omega)$$

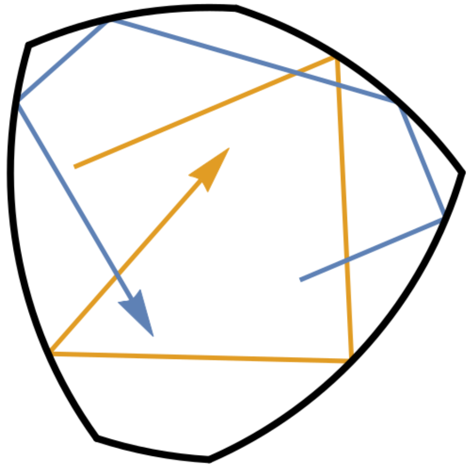
– Hallmark of quantum chaos: **Level repulsion**
Nearest neighbor distr. $p_\beta(s) \sim s^\beta$

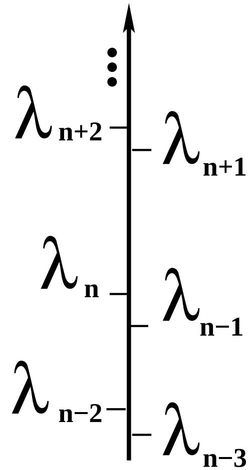
– Chaotic systems fall into **3 symmetry classes**:

$\beta = 1$, GOE: time reversal invariant (TRI)
$\beta = 2$, GUE: broken TRI
$\beta = 4$, GSE: TRI + half integer spin

Uni-directional Systems







Classical: unidirectional (non-ergodic), but chaotic

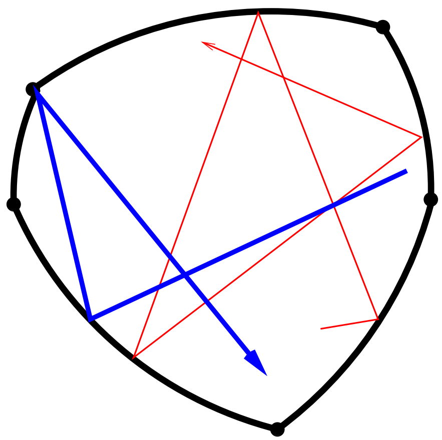
Quantum: both directions “weakly” coupled (by dynamical tunneling, diffraction orbits) $\Rightarrow$

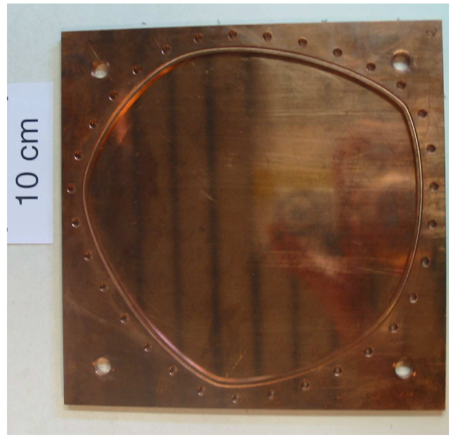
– Quasi-degeneracies

– Anomalous statistics

B.G., *J. Phys. A* **40**, F761 (2007)
B. Dietz, B.G et al., *Phys. Rev. E* **90**, 022903 (2014)

Spectral properties





Collaboration with the experimental group of A. Richter (Darmstadt)

Smooth boundaries:

$$\delta\lambda_n \ll \text{ mean level spacing}$$

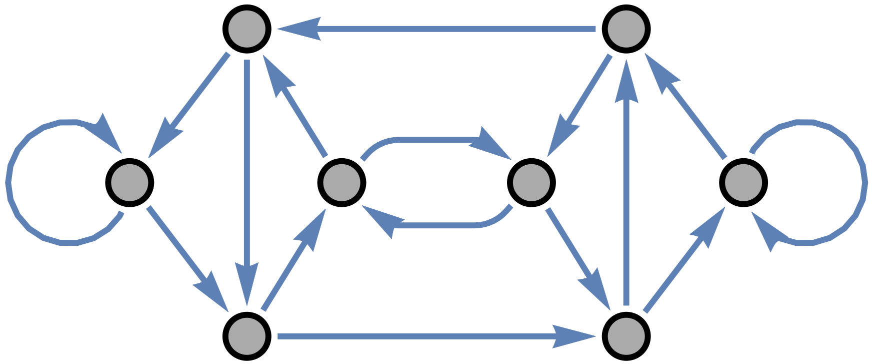
In spite TRI, statistics are of GUE type

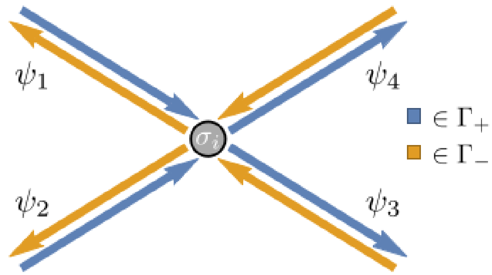
Non-smooth boundaries: Strong tunneling due to diffraction $\Rightarrow$

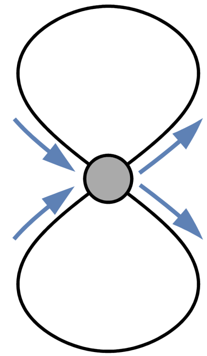
$$\delta\lambda_n \sim \text{ mean level spacing}$$

Anomalous spectral statistics

Uni-directional Quantum Graphs





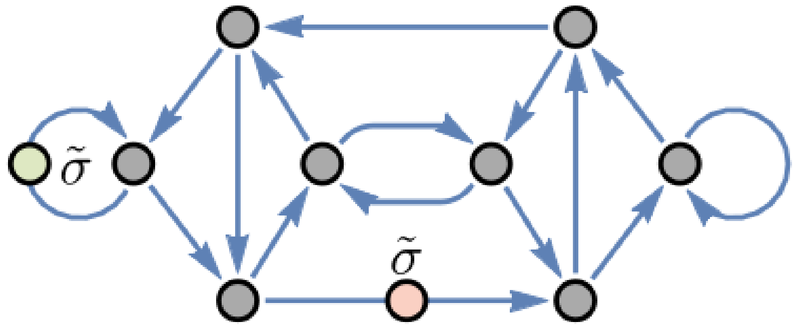


$$\det\left(1 - S\mathcal{L}(k)\right) = 0\,, \quad S = \begin{pmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{S}^T \end{pmatrix}$$

$\mathcal{L}(k) = \text{diag}\left(e^{ikl_1}, \ldots, e^{ikl_{2B}}\right)$, $l_i = l_{i+B}$, $B = \#\text{edges}$

Spectrum of $S\mathcal{L}(k)$ is **doubly degenerate**
Spectral statistics are of GUE type

Adding back-scatterer



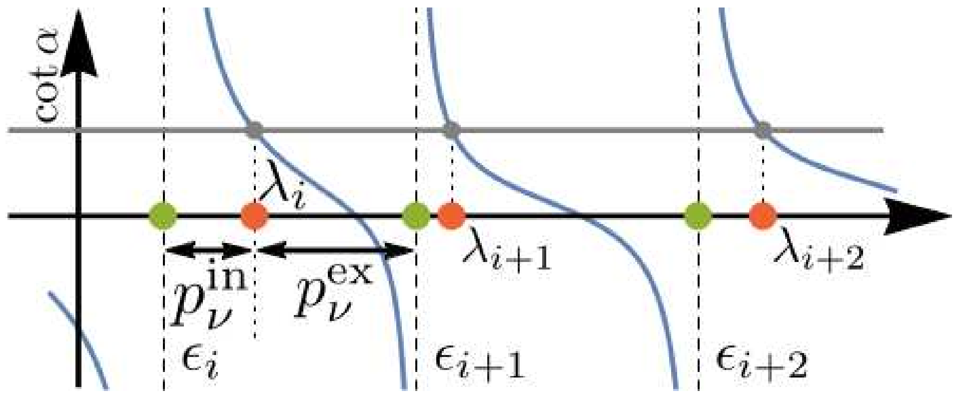
$$\tilde{\sigma} = e^{i\alpha} \begin{pmatrix} i\sin\alpha & \cos\alpha \\ \cos\alpha & i\sin\alpha \end{pmatrix}$$

α controls strength of back-scattering

$\Longrightarrow$ Lifting degeneracies

Q. What is the nearest-neighbor distribution $p(s)$ between eigenvalues?

Adding back-scatterer



Half of the spectrum doesn't change: $\{\epsilon_i\}$
Secular equation for other half: $\{\lambda_i\}$

$$1/\nu \equiv \cot\alpha = \sum_{m=1}^{B} |A_m|^2 \cot\left(\frac{\lambda - \epsilon_m}{2}\right)$$

$|A_m|^2$ = Amplitude of original eigenstates at $\tilde{\sigma}$

Nearest neighbor distribution: $p(s) = \frac{1}{2}\Big(p_{\rm in}(s) + p_{\rm ex}(s)\Big)$

$p_{\rm in}(s)$: distribution of $\epsilon_i - \lambda_i$
$p_{\rm ex}(s)$: distribution of $\lambda_i - \epsilon_{i+1}$

Random Matrix model

RMT assumptions:

$\{\epsilon_n\} \sim$ CUE distributed

$$p(|A_m|^2) = N \exp\left(-N|A_m|^2\right)$$

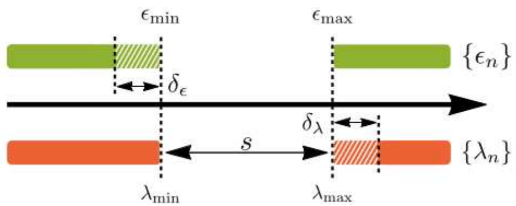
Joint probability:

$$P(\{\epsilon_i\}, \{\lambda_j\}) \propto \left(\prod_{\substack{i,j=1 \\ i>j}}^{N} 4 \sin \frac{\epsilon_i - \epsilon_j}{2} \sin \frac{\lambda_i - \lambda_j}{2} \right) \exp\left(-\frac{N}{2\nu} \sum_{i=1}^{N} (\lambda_i - \epsilon_i) \right).$$

I. L. Aleiner et al., *Phys. Rev. Lett.* **80**, 814 (1998)

Gap-Probability

$$E = \frac{\det F(\epsilon_{\min}, \epsilon_{\max}; \lambda_{\min}, \lambda_{\max})}{\det F(0,0;0,0)}$$



with the $N \times N$ matrix kernel

$$F_{kl} = \int_{-\pi}^{+\pi} d\epsilon \int_{\epsilon}^{+\pi} d\lambda \, e^{-\frac{N}{2\nu}(\lambda-\epsilon)} e^{i(k-1)\epsilon - i\frac{N-1}{2}\epsilon} e^{i(l-1)\lambda - i\frac{N-1}{2}\lambda}$$
$$\times \big(1-\theta(\epsilon-\epsilon_{\min})\theta(\epsilon_{\max}-\epsilon)\big)\big(1-\theta(\lambda-\lambda_{\min})\theta(\lambda_{\max}-\lambda)\big)$$

Allows to write **splitting distribution as derivative**

$$p_{\rm in}(s) \propto \frac{\partial^2}{\partial\epsilon_{\min}\partial\lambda_{\max}} E(\epsilon_{\min}, \epsilon_{\max}; \lambda_{\min}, \lambda_{\max})\Big|_{\substack{\epsilon_{\min}=\lambda_{\min}=-s\pi/N \\ \epsilon_{\max}=\lambda_{\max}=+s\pi/N}}$$

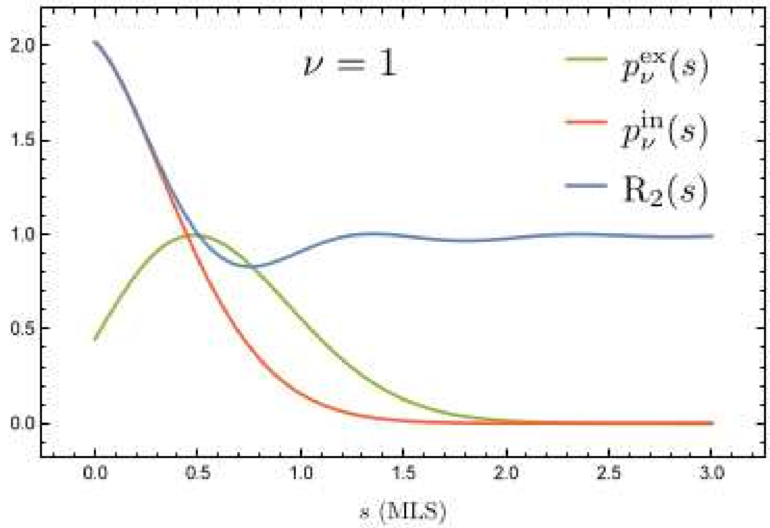
Nearest neighbor distribution

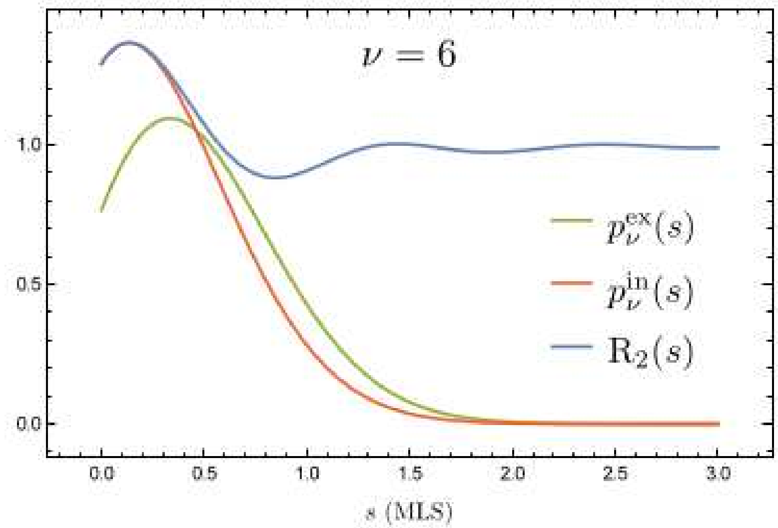
Simple Surmise: Shifted Wigner-Distribution (for $\beta = 2$) gives a good approximation to $p_{\rm in}(s)$, $p_{\rm ex}(s)$:

$$p_{\rm s}(s,c) = \mathrm{p}_{\beta=2}(s-c)/\mathcal{N}(c), \quad \mathcal{N}(c) = \frac{4}{\pi} c\, e^{-\frac{4c^2}{\pi}} + \mathrm{erfc}\left(\frac{2c}{\sqrt{\pi}}\right).$$

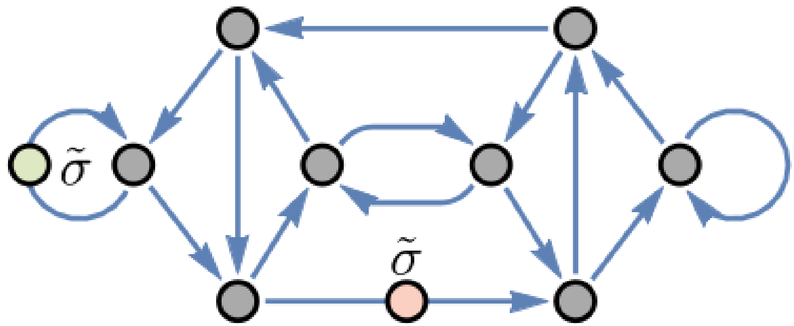
For $p_{\rm in}(s)$ shift c determined by demanding: $p_{\rm s}(0,c) = R_2(0)$

Analytical Results: $p_{\rm in}(s)$, $p_{\rm ex}(s)$ versus 2-point correlator $R_2(s)$



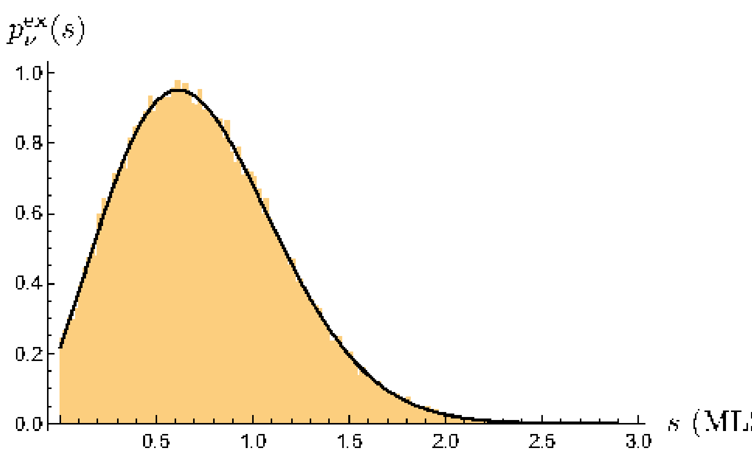


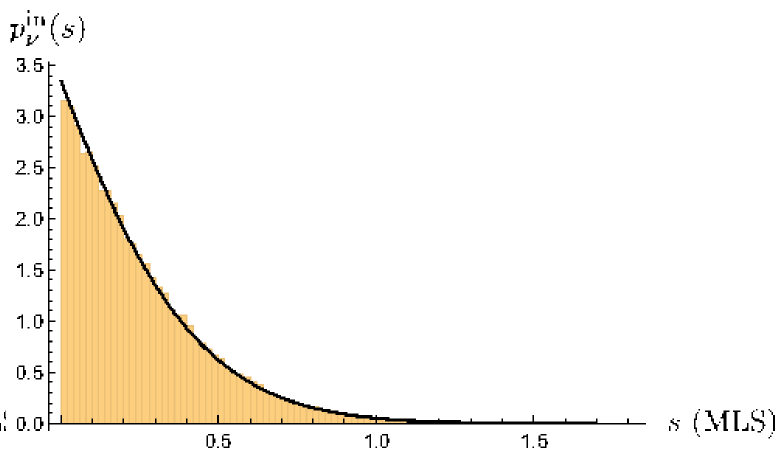
Generic position of scatterer



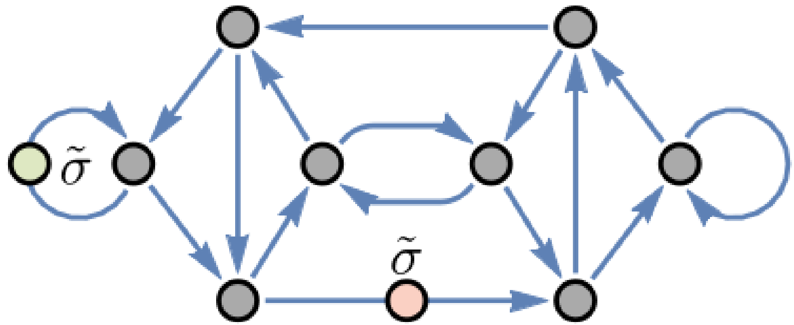
Comparison with Quantum Graphs:

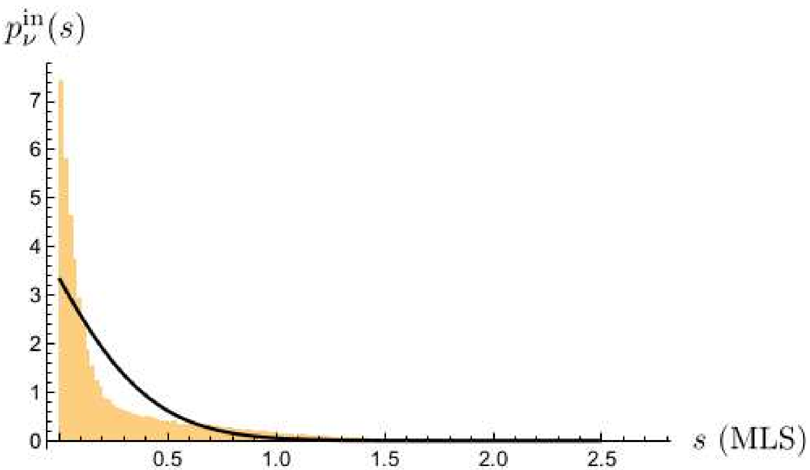
If $\tilde{\sigma}$ sits on “generic” edge $\Longrightarrow$ RMT result holds

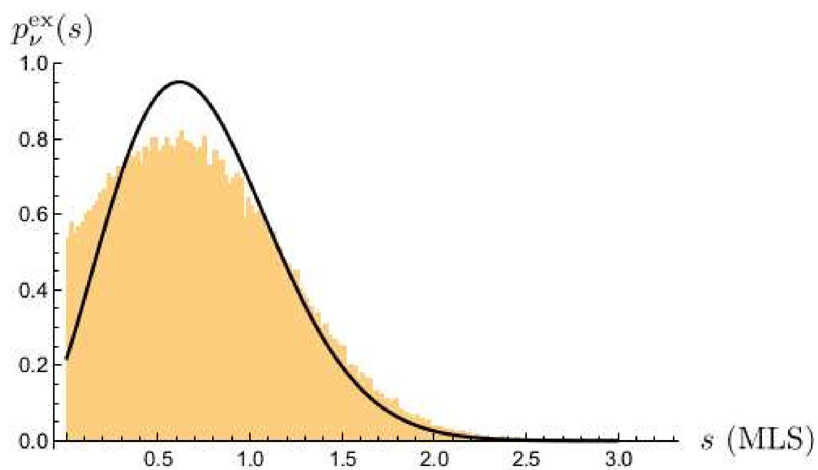




Short loop scatterer





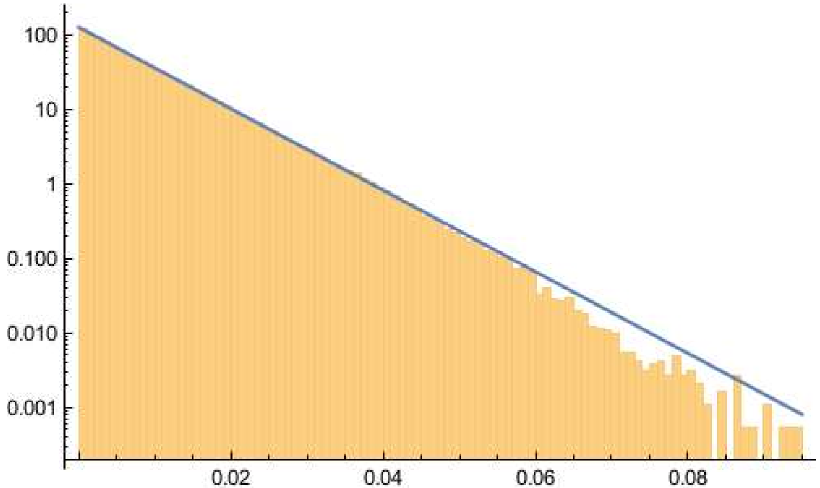


$\tilde{\sigma}$ on short cycle (i.e., self-loop) $\Longrightarrow$ **No RMT result**

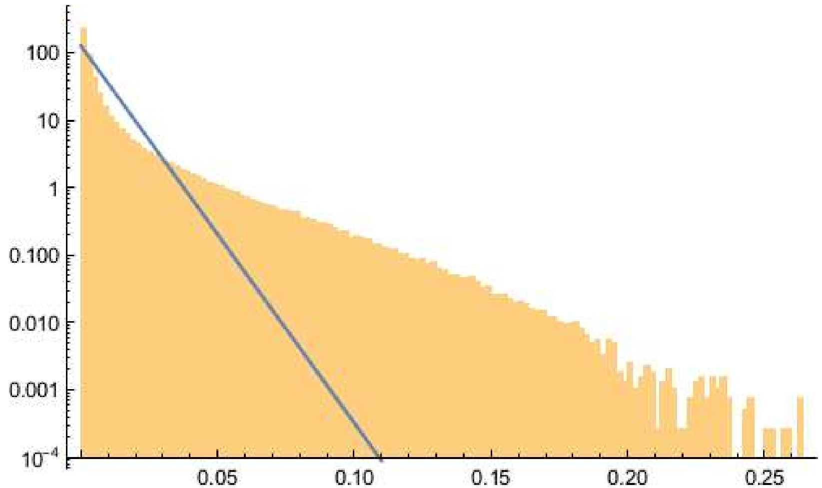
Strong **scarring** of wave-functions on cycle affects $|A_m|^2$ distribution.

Scarring of wave-functions

Deviations from Gaussian statistics



Generic edge



Short loop

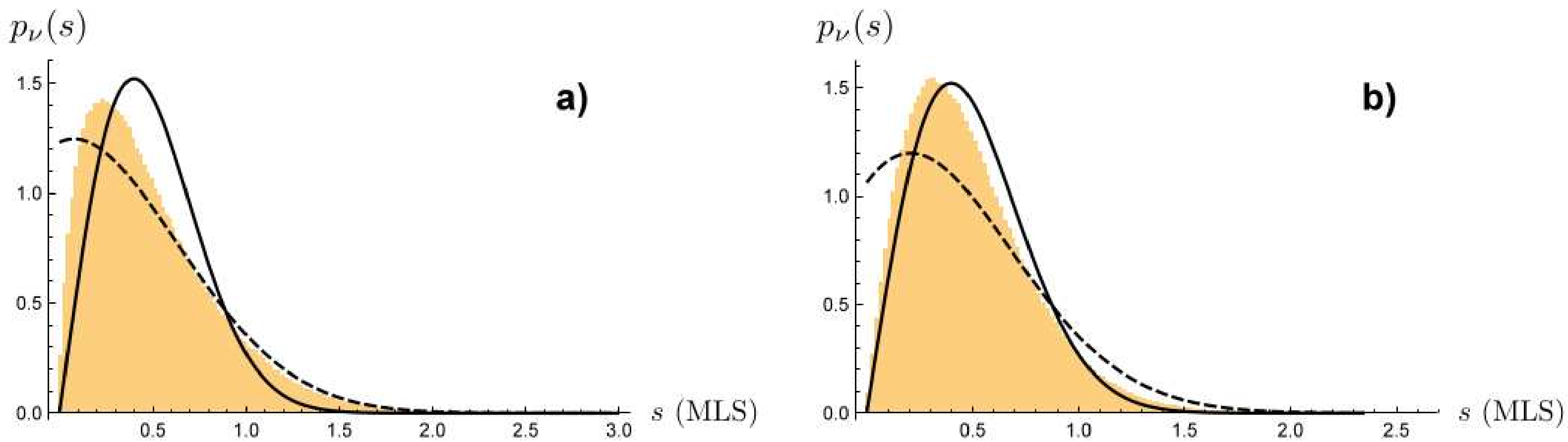
$$P(|\psi_n|^2) \neq N \exp(-|\psi_n|^2 N)$$

Transition GUE → GOE

Higher Rank perturbations:

a) 2 scatterers b) 4 scatterers

Dashed line: 1-rank perturbation, *Solid line:* $p_{\beta=1}(s)$



Only for rank-one perturbation $p(0) \neq 0$, otherwise level repulsion

Breaking unidirectionality $\Longrightarrow$ Fast transition to GOE

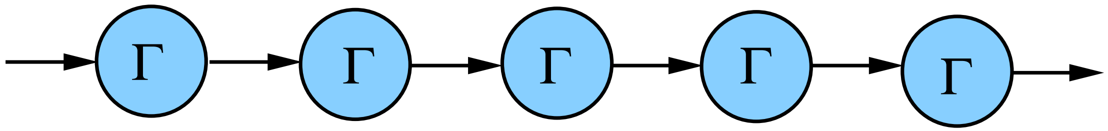
Summary

M. Akila, B.G. J. Phys. A 48, 345101

- Analytic formula for $p(s)$. No level repulsion. Good agreement for generic position of $\tilde{\sigma}$
- No agreement for $\tilde{\sigma}$ positioned on short loops $\Longrightarrow$ Strong scarring
- Fast approach to GOE as $\#$ of scatterers increases
- “Semiclassical” derivation of $R_2(s)$ through **periodic orbit correlations** $\Longleftrightarrow$ Scarring

Another interpretation

Chain of unidirectional graphs Γ



Band structure