Large Deviation Principles for Weakly Interacting Fermions

N. J. B. Aza

Departamento de Física Matemática, Universidade de São Paulo

Joint work with J.-B. Bru, W. de Siqueira Pedra and L. C. P. A. M. Müssnich

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Observe that for a state on the $C^*$-algebra $A$ and $A \in A$ a selfadjoint element, there is a unique probability measure $\mu_{\rho, A}$ on $\mathbb{R}$ such that $\mu_{\rho, A}(\text{spec}(A)) = 1$ and, for all continuous functions $f: \mathbb{R} \rightarrow \mathbb{C}$,

$$\rho(f(A)) = \int_{\mathbb{R}} f(x) \mu_{\rho, A}(dx).$$

$\mu_A = \mu_{\rho, A}$ is the measure associated to $\rho$ and $A$. For a sequence of selfadjoints $\{A_l\}_{l \in \mathbb{R}^+}$ of $A$, and a state $\rho$, we say that these satisfy a Large Deviation Principle (LDP), with scale $|\Lambda_l|$, if, for all Borel measurable $\Gamma \subset \mathbb{R}$,

$$-\inf_{x \in \mathring{\Gamma}} I(x) \leq \liminf_{l \to \infty} \frac{1}{|\Lambda_l|} \log \mu_{A_l}(\Gamma) \leq \limsup_{l \to \infty} \frac{1}{|\Lambda_l|} \log \mu_{A_l}(\Gamma) \leq -\inf_{x \in \Gamma} I(x).$$
Observe that for $\rho$ a state on the $C^*$–algebra $\mathcal{A}$ and $A \in \mathcal{A}$ a selfadjoint element, there is a unique probability measure $\mu_{\rho,A}$ on $\mathbb{R}$ such that $\mu_{\rho,A}(\text{spec}(A)) = 1$ and, for all continuous functions $f : \mathbb{R} \to \mathbb{C}$,

$$\rho(f(A)) = \int_{\mathbb{R}} f(x) \mu_{\rho,A}(dx).$$

$\mu_A \doteq \mu_{\rho,A}$ is the measure associated to $\rho$ and $A$. For a sequence of selfadjoints $\{A_l\}_{l \in \mathbb{R}^+}$ of $\mathcal{A}$, and a state $\rho$, we say that these satisfy a Large Deviation Principle (LDP), with scale $|\Lambda_l|$, if, for all Borel measurable $\Gamma \subset \mathbb{R}$,

$$- \inf_{x \in \Gamma} \mathcal{I}(x) \leq \liminf_{l \to \infty} \frac{1}{|\Lambda_l|} \log \mu_{A_l}(\Gamma) \leq \limsup_{l \to \infty} \frac{1}{|\Lambda_l|} \log \mu_{A_l}(\Gamma) \leq - \inf_{x \in \Gamma} \mathcal{I}(x)$$
Large Deviation Theory and Quantum Lattice Systems

- To find an LDP we desire to use the Gärtner–Ellis Theorem (GET) to \( \mu_{A_l} \), through the scaled cumulant generating function

\[
\bar{f}(s) = \lim_{l \to \infty} \frac{1}{|\Lambda_l|} \log \rho(e^{s|\Lambda_l|A_l}), \quad s \in \mathbb{R}.
\]

- If \( \bar{f} \) exists and is differentiable, then the good rate function \( \mathcal{I} \) is the Legendre–Fenchel transform of \( \bar{f} \).

- In the case of lattice fermions we represent \( \bar{f} \) as a Berezin–integral and analyse it using “tree expansions”. The scale \(|\Lambda_l|\) will be then the volume of the boxes \( \Lambda_l \):

\[
\Lambda_l \doteq \{(x_1, \ldots, x_d) \in \mathbb{Z}^d : |x_1|, \ldots, |x_d| \leq l\} \in \mathcal{P}_f(\mathbb{Z}^d).
\]

- For lattice fermions, \( \mathcal{A} \) is the CAR \( C^* \)-algebra generated by the identity \( 1 \) and \( \{a_{s,x}\}_{s,x \in S} \). \( \mathcal{L} \doteq S \times \mathbb{Z}^d \) where \( S \) is the set of Spins of single fermions. However, our proofs do not depend on the particular choice of \( S \).
Large Deviation Theory and Quantum Lattice Systems

- CAR:

\[ \{a_x, a_{x'}\} = 0, \quad \{a_x, a^*_x\} = \delta_{x,x'} 1. \]

- \( \mathcal{A}_\Lambda \subset \mathcal{A} \) is the \( C^* \)-subalgebra generated \( 1 \) and \( \{a_x\}_{x \in \Lambda} \).

- An interaction \( \Phi \) is a map \( \mathcal{P}_f(\mathbb{Z}^d) \rightarrow \mathcal{A} \) s.t. \( \Phi_\Lambda = \Phi^*_\Lambda \in \mathcal{A}^+ \cap \mathcal{A}_\Lambda \) and \( \Phi_\emptyset = 0 \).

- \( \Phi \) is of finite range if for \( \Lambda \in \mathcal{P}_f(\mathbb{Z}^d) \) and some \( R > 0, \text{diam} \ Lambda > R \rightarrow \Phi_\Lambda = 0 \).

- For any interaction \( \Phi \), we define the space average \( K_\Lambda^\Phi \in \mathcal{A}_\Lambda \) by

\[
K_\Lambda^\Phi = \frac{1}{|\Lambda_i|} \sum_{\Lambda \in \mathcal{P}_f(\mathbb{Z}^d), \Lambda \in \Lambda_i} \Phi_\Lambda.
\]
Main Result

Note that finite range interactions define equilibrium (KMS) states of $\mathcal{A}$.

**Theorem (A., Bru, Müssnich, Pedra)**

Let $\beta > 0$ and consider any finite range translation invariant interaction $\Psi = \Psi_0 + \Psi_1$. If the interparticle component $\Psi_1$ ($\Psi_0$ is the free part) is small enough (depending on $\beta$), then any invariant equilibrium state $\rho$ of $\Psi$ and the sequence of averages $K_l^\Phi$ of ANY translation invariant interaction $\Phi$, have an LDP and $s \mapsto \bar{f}(s)$ is analytic at small $s$. 
Main Result

Remarks

1. Note that, in contrast to previous results, we do not impose $\beta$ to be small or $\Phi$ (defining $K_i^\Phi$) to be an one–site interaction.

2. Uniqueness of KMS states is not used.


4. Determinant bounds or study of Large Determinants.

5. Direct representation of $\bar{f}$ by Berezin–integrals. In particular we do not use the correlation functions.

6. Beyond the LDP, the analyticity of $\bar{f}(\cdot)$ together with the Bryc Theorem implies the Central Limit Theorem for the system.
Main Result

Sketch of the proof.

\[
\bar{f}(s) = \lim_{l \to \infty} \lim_{l' \to \infty} \frac{1}{\Lambda_l} \log \frac{\text{tr}(e^{-\beta H_{l'}} e^{sK_l})}{\text{tr}(e^{-\beta H_{l'}})}.
\]
Main Result

Sketch of the proof.

1. \[ \bar{f}(s) = \lim_{l \to \infty} \lim_{l' \to \infty} \frac{1}{|\Lambda_l|} \log \frac{\text{tr}(e^{-\beta H_{l'}} e^{s K_l})}{\text{tr}(e^{-\beta H_{l'}})}. \]

2. From a Feynmann–Kac–like formula for traces, we write the KMS state as a Berezin–integral

\[
\frac{\text{tr}_{\Lambda^* \mathcal{D}}(e^{-\beta H_{l'}} e^{s K_l})}{\text{tr}_{\Lambda^* \mathcal{D}}(e^{-\beta H_{l'}^{(0)}})} = \lim_{n \to \infty} \int d\mu_{C_{l''}^{(n)}}(\mathcal{D}^{(n)}) e^{\mathscr{H}_{l,l''}^{(n)}}.
\]
Main Result

Sketch of the proof.

1  \[ f(s) = \lim_{l \to \infty} \lim_{l' \to \infty} \frac{1}{|\Lambda|} \log \frac{\text{tr}(e^{-\beta H_{l'}} e^{s K_l})}{\text{tr}(e^{-\beta H_{l'}})}. \]

2  From a Feynmann–Kac–like formula for traces, we write the KMS state as a Berezin–integral

\[ \frac{\text{tr}_{\wedge^* S_3}(e^{-\beta H_{l'}} e^{s K_l})}{\text{tr}_{\wedge^* S_3}(e^{-\beta H_{l'}^{(0)}})} = \lim_{n \to \infty} \int d\mu_{\mathcal{C}_l^{(n)}}(S_3^{(n)}) e^{\mathcal{W}_{l, l'}.} \]

3  The covariance \( C_l^{(n)} \) satisfies:

\[ \left| \det \left[ (\varphi_a^{(k_a)}) C_l^{(n)} (\varphi_b^{(k_b)}) \right]_{a,b=1}^{m} \right| \leq \left( \prod_{a=1}^{m} \| \varphi_a^* \|_{S_3^*} \right) \left( \prod_{b=1}^{m} \| \varphi_b \|_{S_3} \right). \]
Main Result

Sketch of the proof.

1. \( \bar{f}(s) = \lim_{l \to \infty} \lim_{l' \to \infty} \frac{1}{|\Lambda_l|} \log \frac{\text{tr}(e^{-\beta H_{l'}} e^{sK_l})}{\text{tr}(e^{-\beta H_{l'}})}. \)

2. From a Feynmann–Kac–like formula for traces, we write the KMS state as a Berezin–integral

\[
\frac{\text{tr}^\wedge_* \delta_j(e^{-\beta H_{l'}} e^{sK_l})}{\text{tr}^\wedge_* \delta_j(e^{-\beta H_{l'}^{(0)}})} = \lim_{n \to \infty} \int d\mu C^{(n)}_{\mu} (\delta_j^{(n)}) e^{\mathcal{W}_l^{(n)}}.
\]

3. The covariance \( C^{(n)}_{\mu} \) satisfies:

\[
\left| \det \left[ \left( \varphi^*_a \right)^{(k_a)} \left( C^{(n)}_{\mu} \left( \varphi^{(k_b)}_b \right) \right) \right]_{a,b=1}^m \right| \leq \left( \prod_{a=1}^m \| \varphi^*_a \|_{\delta_j^*} \right) \left( \prod_{b=1}^m \| \varphi_b \|_{\delta_j} \right).
\]

Use Brydges–Kennedy Tree expansions (BKTE) to verify GET. BKTE are solution of an infinite hierarchy of coupled ODEs...
Perspectives and Questions

Perspectives:

1. Quantum Hypothesis Testing? Open problems, e.g., study thermodynamic limit of the relative entropy between equilibrium state $\omega^\beta_\Lambda \in \mathcal{A}_\Lambda$ and translation invariant state $\omega_\Lambda$.
2. Related problems to our approach.
3. ...
Perspectives and Questions

Perspectives:

1 Quantum Hypothesis Testing? Open problems, e.g., study thermodynamic limit of the relative entropy between equilibrium state $\omega^\beta_\Lambda \in \mathcal{A}_\Lambda$ and translation invariant state $\omega_\Lambda$.

2 Related problems to our approach.

3 . . .

Open Questions:

1 LDP for time correlation (transport coefficients)?

2 Systems in presence of disorder?

3 What about LDP for commutators of averages $i[K^{\Phi_1}, K^{\Phi_2}]$ in place of simple averages $K^{\Phi}$? (Also related to transport)

4 . . .
Thank you!
Supporting facts

1. For any invertible operator $C \in \mathcal{B}(\mathfrak{H})$ and $\xi \in \wedge^* (\mathfrak{H} \oplus \tilde{\mathfrak{H}})$, the Gaussian Grassmann integral: $\int d\mu_C (\mathfrak{H}) : \wedge^* (\mathfrak{H} \oplus \tilde{\mathfrak{H}}) \rightarrow \mathbb{C} \mathbf{1}$ with covariance $C$, is defined by

$$\int d\mu_C (\mathfrak{H}) \xi = \det (C) \int d (\mathfrak{H}) e^{\langle \mathfrak{H}, C^{-1} \mathfrak{H} \rangle} \wedge \xi.$$ 

2. $\int d\mu_C (\mathfrak{H}) \mathbf{1} = \mathbf{1}$ and for any $m, n \in \mathbb{N}$ and all $\psi_1, \ldots, \psi_m \in \mathfrak{H}$, $\varphi_1, \ldots, \varphi_n \in \mathfrak{H}$,

$$\int d\mu_C (\mathfrak{H}) \psi_1 \cdots \psi_m \varphi_1 \cdots \varphi_m = \det [\varphi_k (C \varphi_l)]_{k,l=1}^m \delta_{m,n} \mathbf{1}$$

3. For all $N \in \mathbb{N}$ and $A_0, \ldots, A_{N-1} \in \mathcal{B}(\wedge^* \mathfrak{H})$,

$$\text{Tr}_{\wedge^* \mathfrak{H}} (A_0 \cdots A_{N-1}) \mathbf{1} = \left( \prod_{k=0}^{N-1} \int d (\mathfrak{H}^{(k)}) \right) E_{\mathfrak{H}}^{(N)} \left( \prod_{k=0}^{N-1} \varphi^{(k)} (A_k) \right),$$

where $E_{\mathfrak{H}}^{(N)} = e^{\langle \mathfrak{H}^{(0)}, \mathfrak{H}^{(0)} \rangle + \langle \mathfrak{H}^{(0)}, \mathfrak{H}^{(N-1)} \rangle + \sum_{k=1}^{N-1} (\langle \mathfrak{H}^{(k)}, \mathfrak{H}^{(k)} \rangle - \langle \mathfrak{H}^{(k)}, \mathfrak{H}^{(k-1)} \rangle)}$, 

$\varphi^{(k)} = \varphi^{(k,0)} \circ \varphi : \mathcal{B}(\wedge^* \mathfrak{H}) \rightarrow \wedge^* (\mathfrak{H}^{(k)} \oplus \tilde{\mathfrak{H}}^{(k)})$ and for $i, j, k, l \in \{0, \ldots, N\}$, $\varphi^{(k,l)} : \wedge^* (\mathfrak{H}^{(i)} \oplus \tilde{\mathfrak{H}}^{(j)}) \rightarrow \wedge^* (\mathfrak{H}^{(k)} \oplus \tilde{\mathfrak{H}}^{(l)}).$