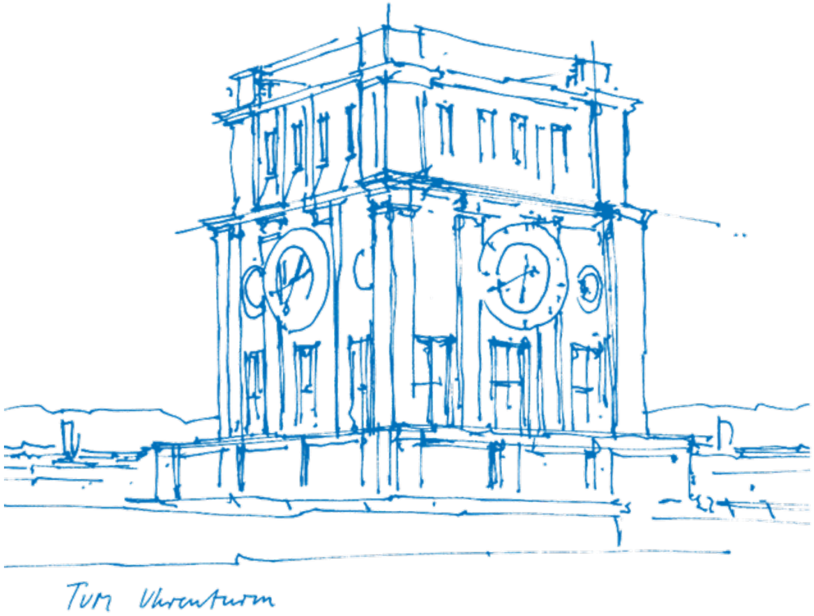
Localization in the Hierarchical Anderson Model

Per von Soosten

Joint work with Simone Warzel

Based on arXiv: 1608.01602



Mathematical Results in Quantum Physics 13

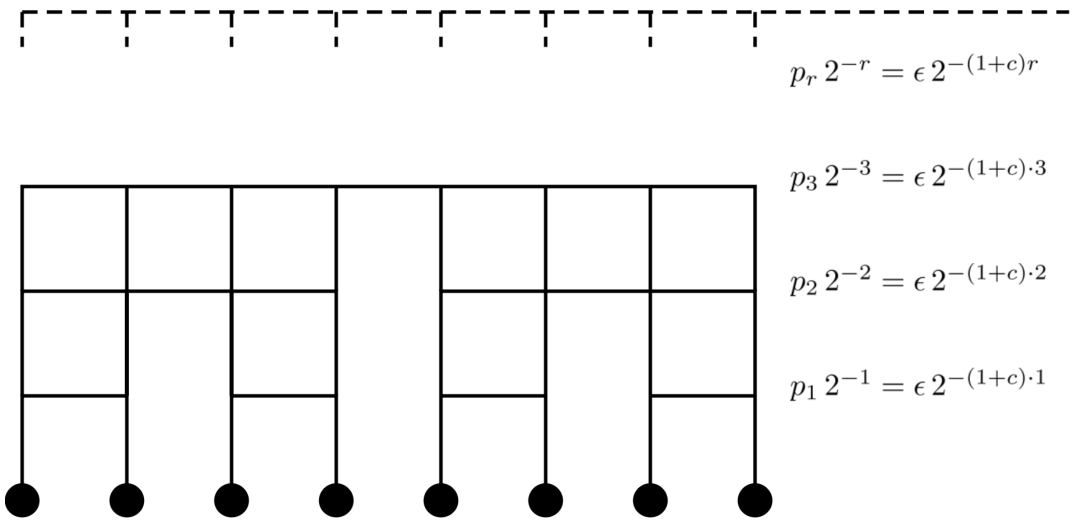
Georgia Institute of Technology

10/09/2016

DFG Deutsche Forschungsgemeinschaft

The Hierarchical Anderson Model

- We will consider $H = \Delta + V$ on $\ell^2(\mathbb{N})$
- $V = \sum V_k |\delta_k\rangle\langle\delta_k|$ is a random potential with $\{V_k\}$ drawn independently from $\varrho \in L^\infty$
- Δ represents hierarchically organized hopping on $\mathbb{N}$:



Hierarchical hopping characterized by $p_r = \epsilon 2^{-cr}$ with $c > 0$ and $\epsilon > 0$

- Induced metric $d(j,k) = \inf\{r : j, k$ lie in common cluster of size $2^r\}$

The Hierarchical Laplacian Δ

- Infinitely degenerate eigenvalues

$$\lambda_s = \sum_{r=1}^{s} p_r$$

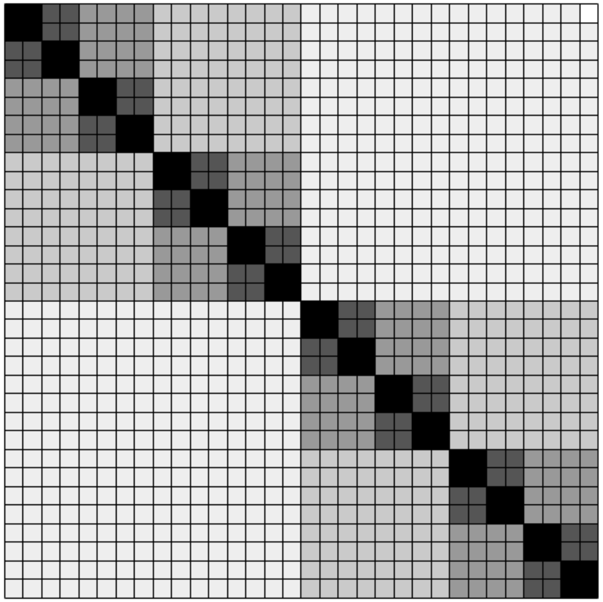
accumulating at $\lambda_\infty = \sum p_r$

- Eigenfunctions delocalize as $\lambda_s \to \lambda_\infty$, e.g.

$$|\operatorname{supp} \psi_s| = 2^s$$

- Defines an effective spectral dimension

$$d_s := \lim_{t \to 0} \frac{\log \nu(\lambda_\infty - t, \lambda_\infty)}{\log t} = \frac{2}{c}$$



Heat map of Δ

Motivation and Previous Work

- Retaining crucial features while gaining simplicity
  Ising ferromagnets (Dyson 1969)
  $|\Phi|^4$ model (Gawędzki, Kupiainen 1983)
  Spin glasses (Gardner 1984)
  Directed polymers (Derrida, Griffiths 1989)
  ...

- Predictions about localization-delocalization transition
  Critical dimension $d_s > 4$ (Bovier 1990)
  Pure-point spectrum when ϱ is Cauchy (Molchanov 1996)
  Pure-point spectrum when $d_s < 4$ (Kritchevski 2007)
  Poisson statistics when $d_s < 1$ (Kritchevski 2008)
  Critical energy at $\sum p_r$ when $d_s > 2$ (Metz, Leuzzi, Parisi, Sacksteder 2014)

Assumption on the Disorder Density ϱ

Let $T_p\varrho$ denote the density of

$$\left(\frac{1}{2V}+\frac{1}{2V'}\right)^{-1}+p$$

where V and V' are independently drawn from ϱ.

Assumption

$I \subset \mathbb{R}$ is an interval for which there exists $\delta > 0$ with the property that

$$\|T_{p_r}\dots T_{p_1}\varrho_E\|_\infty = \mathcal{O}\left(2^{(c-\delta)r}\right)$$

uniformly in $E \in I$.

Valid with $I = \mathbb{R}$ if:

- ϱ has a Cauchy component
- ϱ is Gaussian and $d_s < 4$, or
- $d_s < 2$.

Numerical studies show that the assumption is true whenever $\varrho > 0$.

Main Results

Theorem

- *There exist* $\mu > 0$ *and* $K < \infty$ *such that*

$$\sup_{n \geq 1} \sum_{k \in \mathbb{N}} 2^{\mu d(0,k)} \mathbb{E}|G_n(0,k;E)|^s \leq K$$

  uniformly in $E \in I$*.*
- *The rescaled eigenvalue point process*

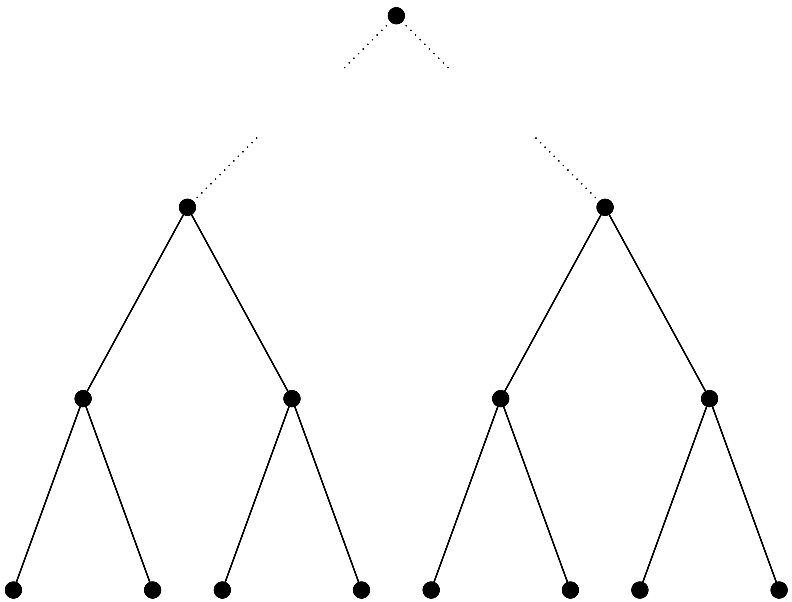
$$\mu_n(f) = \sum_{\lambda \in \sigma(H_n)} f(2^n(\lambda - E)), \qquad f \in C_0(\mathbb{R})$$

  converges in distribution to a Poisson point process with intensity $\nu(E)$*.*

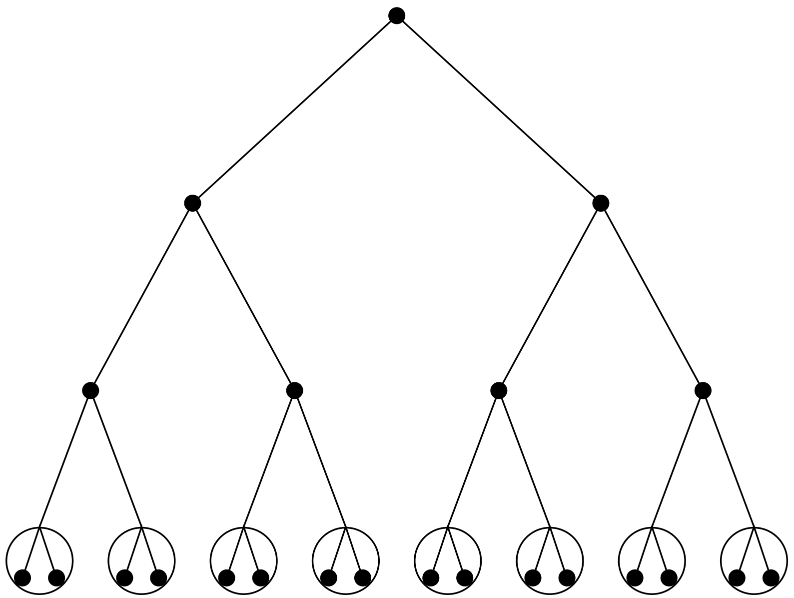
Remarks

- *Implies both exponential spectral localization and dynamical localization in the mean*
- *Exponential spectral localization can also be proved without any assumption on the density* ϱ
- *This extends results of Molchanov (1996) and Kritchevski (2007)*

Two Philosophies



Philosophy I: Removing the root



Philosophy II: Collapsing the leaves

$$H_n = H_{n-1} \oplus H'_{n-1} + p_n|\varphi_n\rangle\langle\varphi_n|$$

FKS Renormalization

The FKS Renormalization

- Recall that H depends on $p_r = \epsilon 2^{-cr}$ and ϱ.
- Renormalization Dynamics:

$$\mathcal{R}\left((p_r)_{r=1}^{\infty}, \varrho\right) = \left(\{p_{r+1}\}_{r=1}^{\infty}, T_{p_1}\varrho\right)$$

- Formula of Metz, Leuzzi, Parisi & Sacksteder (2014):

$$G_n(0, 2k; 0) = 2\frac{V_1}{V_0 + V_1}\frac{V_{2k+1}}{V_{2k} + V_{2k+1}}(\mathcal{R}G)_{n-1}(0, k; 0)$$

- Proof idea: Take the Feshbach-Krein-Schur complement in an appropriate basis.

Driving Towards High Disorder

- FKS renormalization decreases the hopping strength $\mathcal{R}p_r = p_{r+1} = 2^{-c}p_r$.
- By assumption $\mathcal{R}\varrho = T_{p_1}\varrho$ does not decrease the disorder strength.
- After a **finite** number of steps an Aizenman-Molchanov type high disorder argument applies, i.e.,

$$\mathbb{E}\left|\mathcal{R}^N G_{n-N}\left(0, \lfloor k/2^N \rfloor, 0\right)\right|^s \le C_N\, 2^{-\mu d(0,k)}.$$

Key Lemma

There exists a constant $C_s(\varrho) < \infty$ such that

$$\mathbb{E}\,|G_n(0,k;0)|^s \le C_s(\varrho)\mathbb{E}\,|\mathcal{R}G_{n-1}(0,\lfloor k/2 \rfloor;0)|^s$$

for all $k \in B_n \setminus \{0\}$.

Summary

- The hierarchical Anderson model exhibits eigenfunction-correlator localization and Poissonian level statistics at all energies in all spectral dimensions for any strength of the disorder.
- Renormalization methods reduce strong localization phenomena to a classical real analysis problem.
- Open Problem: Suppose that $\varrho \in C^1$ is a strictly positive probability density. Prove that for every $E_0 \in \mathbb{R}$ there exists a small interval I about E_0 such that

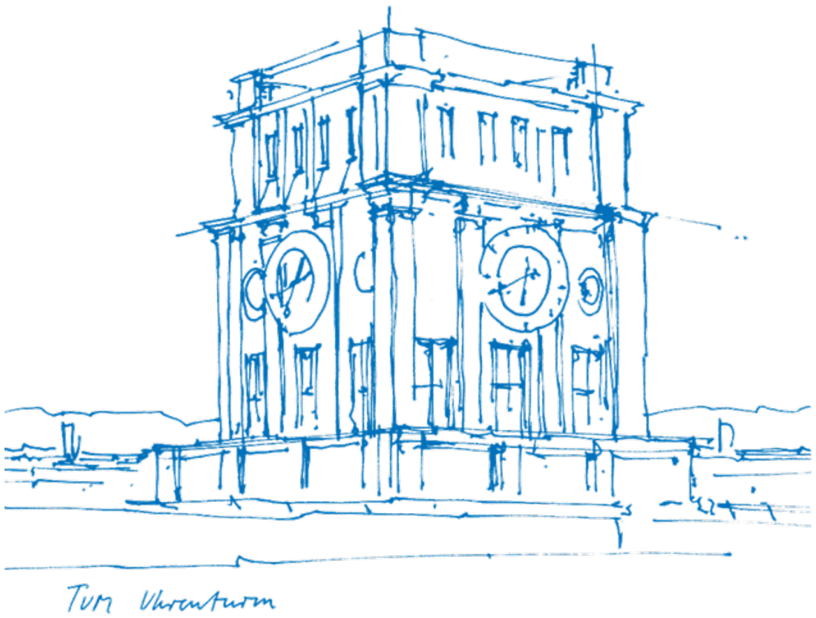
$$\| T_{p_r} \dots T_{p_1} \varrho_E \|_\infty = \mathcal{O}\left(2^{(c-\delta)r} \right)$$

uniformly in $E \in I$.

Thank You!

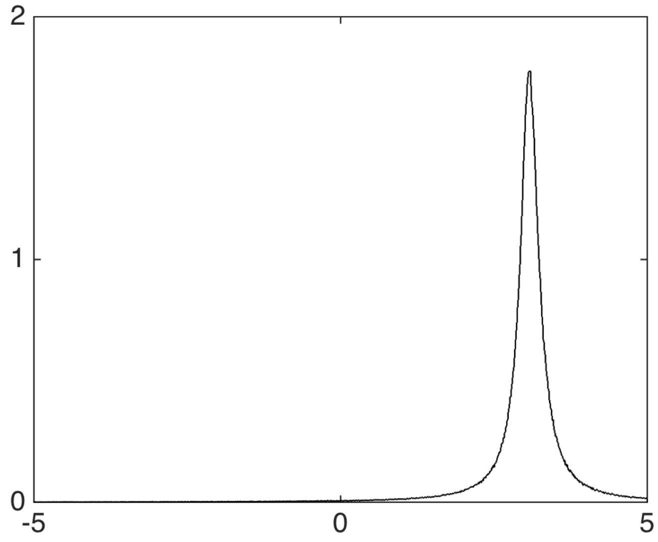
Further Information:

P. von Soosten and S. Warzel. *Renormalization Group Analysis of the Hierarchical Anderson Model.* arXiv: 1608.01602

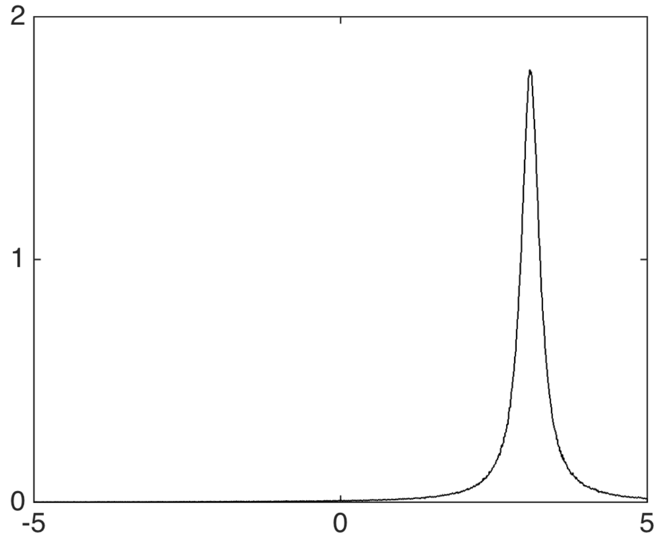


Numerical Evidence

- If $\varrho(0) > 0$, $\left(\frac{1}{2V} + \frac{1}{2V'}\right)^{-1}$ does not concentrate since V^{-1} has only fractional moments
- In particular, $T_p\varrho$ is in an α-stable basin of attraction (with $\alpha = 1$)



$T_{p_5} \dots T_{p_1}\varrho$ when $\varrho = \mathcal{N}(0, 1/4)$ and $c = 1/3$



Cauchy distribution with $\mu = 3.09$ and $\sigma = 0.18$

Poisson Statistics: Sketch of the Proof

- Remove the most extended hopping $H_n = H_{n-1} \oplus H'_{n-1} + p_n|\varphi_n\rangle\langle\varphi_n|$
- The spectral shift is controlled by $F(E) = \langle\varphi_n, (H_{n-1} \oplus H'_{n-1} - E - i0)^{-1}\varphi_n\rangle$
- FKS calculations show that $1/F(E)$ is distributed according to $T_{p_n} \dots T_{p_1}\varrho_E$
- Thus μ_n is well-approximated by $\mu_{n-1} + \mu'_{n-1}$
- Iterating this shows that μ is infinitely divisible