Localization in the disordered Holstein Model

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Context: Many Body Localization

What is many body localization?

- This is an active area of research in physics.
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- Not many mathematical results
  - John Imbrie's result for disordered spin chains is the notable exeption
- **I'm not going to address MBL in this talk**

(But it is the context….)

(One Particle) Holstein Model

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$$H_{\mathrm{Hol}}^{(\Lambda)} = \gamma \Delta^{(\Lambda)} + \omega (b_{\mathbf{X}}^\dagger - \beta^*)(b_{\mathbf{X}} - \beta) + \omega \sum_{\substack{x \in \Lambda \\ x \neq \mathbf{X}}} b_x^\dagger b_x$$

Properties of the 1P Holstein Model

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- $\mathcal{H}_\Lambda \;=\; \{\psi : \Lambda \to \mathcal{F}_\Lambda\} \;=\; \ell^2(\Lambda; \mathcal{F}_\Lambda)$
- $\Delta^{(\Lambda)}\psi(x) \;=\; \sum_{\substack{x \sim y \\ y \in \Lambda}} \psi(x) - \psi(y)$
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- ω small: the picture is much less clear

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- $H_\gamma^{(\Lambda)}$ is non-negative definite.
- Spectrum is contained in

$$\bigcup_n [\omega n, \omega n + V_+ + 4d\gamma]$$

Main Result

Theorem: *For each n there is γ_n such that if $\gamma < \gamma_n$, then the eigenstates in the n-th band of the spectrum are exponentially localized in position and localized in a suitable metric in Fock space.*

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Fock Space Displacement Operators

- Consider the Hilbert space for a single oscillator:

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- Let $D_\beta = \mathrm{e}^{\beta b^\dagger - \beta^* b}$
- This operator is **unitary** and **intertwines** the eigenbasis for $b^\dagger b$ with that for $(b^\dagger - \beta^*)(b - \beta)$:

$$(b^\dagger - \beta^*)(b - \beta) D_\beta |m\rangle = D_\beta b^\dagger b |m\rangle = m D_\beta |m\rangle$$

Displacement Operator Bounds

Proposition: *Let* $\mu > 0$. *Then there is a finite constant* $A = A_{\mu,\beta}$ *such that*

$$|\langle m| D_\beta |n\rangle| \leqslant A\mathrm{e}^{-\mu|\sqrt{n}-\sqrt{m}|}$$

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- It follows from the following remarkable identity

$$\sum_m \mathrm{e}^{2\mu(m-n)} \left|\langle m|\, D_\beta\, |n\rangle\right|^2 = \mathrm{e}^{(\mathrm{e}^{2\mu}-1)|\beta|^2} L_n\left(-|\beta|^2 (\mathrm{e}^{\mu} - \mathrm{e}^{-\mu})^2\right)$$

where L_n is the n-th order Laguerre polynomial.

Transforming the basis

$$H_0^{(\Lambda)} = \omega H_{\text{ph}}^{(\Lambda)} + V^{(\Lambda)}$$

$$H_{\text{ph}}^{(\Lambda)} = \sum_{x\in\Lambda} a_x^\dagger a_x \qquad\qquad a_x = b_x - \beta I[\boldsymbol{X} = x]$$

$$D_\beta^{(x)} |\boldsymbol{m}\rangle := \mathrm{e}^{\beta b_x^\dagger - \beta^* b_x} |\boldsymbol{m}\rangle$$

$$H_0^{(\Lambda)} |x, \boldsymbol{m}\rangle = (\omega|\boldsymbol{m}| + v_x) |x, \boldsymbol{m}\rangle$$

$$|x, \boldsymbol{m}\rangle := |x\rangle \otimes D_\beta^{(x)} |\boldsymbol{m}\rangle$$

Kinetic Operator

$$\langle x, \boldsymbol{m}| \Delta^{(\Lambda)} |y, \boldsymbol{\xi}\rangle$$
$$= \begin{cases} 2d & x = y \ \& \ \boldsymbol{m} = \boldsymbol{\xi} \\ \langle \boldsymbol{m}(x)| D_{-\beta} |\boldsymbol{\xi}(\boldsymbol{x})\rangle \langle \boldsymbol{m}(y)| D_\beta |\boldsymbol{\xi}(\boldsymbol{y})\rangle & x \sim y \ \& \ \boldsymbol{m}(u) = \boldsymbol{\xi}(u) \text{ for } u \neq x, y \\ 0 & \text{otherwise.} \end{cases}$$

- Matrix elements decay off the diagonal

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This leads to Combes-Thomas type estimate for

$$H_\gamma^{(\Lambda)} = H_0^{(\Lambda)} + \gamma \Delta^{(\Lambda)}$$

in this basis.

Main Result revisited

Theorem: *For each* n *there is* γ_n *such that if* $\gamma < \gamma_n$, *then*

$$\mathbb{E}\left(|G(x,\boldsymbol{m},y,\boldsymbol{\xi})|^s\right) \leqslant A\mathrm{e}^{-s\nu D(x,\boldsymbol{m},y,\boldsymbol{\xi})}\mathrm{e}^{-\mu s|\sqrt{\boldsymbol{m}}-\sqrt{\boldsymbol{\xi}}|}$$

for energies in the n-th band where $\nu,\ \mu > 0$, $s < 1$ *and*

$$D(x,\boldsymbol{m},y,\boldsymbol{\xi}) = |x-y| + R(\boldsymbol{m},\boldsymbol{\xi})$$

with $R(\boldsymbol{m},\boldsymbol{\xi})$ *a measure of the size of the set on which* $\boldsymbol{m}$ *and* $\boldsymbol{\xi}$ *differ.*

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- A state $|x, \boldsymbol{m}\rangle$ has on-site energy above the band, unless the oscillators are all in their ground state.
    - The oscillator at x must be in its **deformed** ground state.
- We use a fractional moment method and the Combes-Thomas bound is used to control contributions from the higher bands.

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- Now consider the second band.
- In this band, one of the oscillators can be in it's first excited state.
- In order for this excited oscillator to move, the particle must visit the excited site. **This leads to extra decay if the oscillator states differ in the Green's function.**
- Higher bands are similar (but complicated).

Perspectives and Comments

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- This is **NOT** a fully many body problem.
- But the Hilbert space has features of the many body Hilbert space.
- Spectral localization from eigenfunction correlators.
- Can use spins in place of the oscillators (actually it's technically simpler).

Open Problems

- Improve the dependence of the critical hopping strength on band number (currently super exponential)

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- Randomizing the oscillator frequencies should help. Why doesn't it lead to technical help?
- Could it be that all states are localized in 1D or for weak hopping?

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- What about positive energy density?
  - Do we even need on-site randomness?
- What about a multi/many particle Holstein model?
  - Maybe we could do finitely many particles a la Aizenman, Warzel or Sukhov, Chulaensky

THANK YOU!