KdV equation with almost periodic initial data

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Consider the initial value problem for the KdV equation:

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0$$

$$u(x,0) = V(x)$$

Theorem (McKean–Trubowitz 1976)

If $V \in H^n(\mathbb{T})$, then there is a global solution $u(x,t)$ on $\mathbb{T} \times \mathbb{R}$ and this solution is $H^n(\mathbb{T})$-almost periodic in t.

This means that $u(\cdot,t) = F(\zeta t)$ for some continuous $F : \mathbb{T}^\infty \to H^n(\mathbb{T})$ and $\zeta \in \mathbb{R}^\infty$.

Solutions on $\mathbb{T}$ are periodic solutions on $\mathbb{R}$, which motivates the following:

Conjecture (Deift 2008)

If $V : \mathbb{R} \to \mathbb{R}$ is almost periodic, then there is a global solution $u(x,t)$ that is almost periodic in t.

Even short time existence of solutions is not known in this generality.

Global existence, uniqueness, and almost periodicity

The following theorem solves Deift's conjecture under certain assumptions:

Theorem (Binder–Damanik–Goldstein–Lukic)

If $V : \mathbb{R} \to \mathbb{R}$ is almost periodic, $H_V = -\partial_x^2 + V$ has $\sigma_{\mathrm{ac}}(H_V) = \sigma(H_V) = S$, and S is "thick enough", then

1. *(existence) there exists a global solution $u(x,t)$;*
2. *(uniqueness) if $\tilde{u}$ is another solution on $\mathbb{R} \times [-T, T]$, and*
$$\tilde{u}, \partial_x^3 \tilde{u} \in L^\infty(\mathbb{R} \times [-T, T]),$$
then $\tilde{u} = u$;
3. *(x-dependence) for each t, $x \mapsto u(x,t)$ is almost periodic in x;*
4. *(t-dependence) $t \mapsto u(\cdot, t)$ is $W^{4,\infty}(\mathbb{R})$-almost periodic in t.*

Thickness conditions will be described below.

Application to quasi-periodic initial data

An explicit class of almost periodic initial data covered by this result is the following.

- Consider a quasi-periodic potential given by

$$V(x) = U(\omega x)$$

  with sampling function $U : \mathbb{T}^\nu \to \mathbb{R}$ and frequency vector $\omega \in \mathbb{R}^\nu$.
- Assume that the sampling function is small and analytic:

$$U(\theta) = \sum_{m \in \mathbb{Z}^\nu} c(m) e^{2\pi i m\theta}$$

$$|c(m)| \leq \varepsilon e^{-\kappa_0 |m|}$$

  for some $\varepsilon > 0$, $0 < \kappa_0 \leq 1$.
- We also assume that the frequency vector $\omega \in \mathbb{R}^\nu$ is Diophantine,

$$|m\omega| \geq a_0 |m|^{-b_0}, \quad m \in \mathbb{Z}^\nu \setminus \{0\}$$

  for some $0 < a_0 < 1, \quad \nu < b_0 < \infty$.

Then the above theorem applies as long as $\varepsilon < \varepsilon_0(a_0, b_0, \kappa_0)$.

Application to quasi-periodic initial data

Theorem

If V is quasi-periodic with a Diophantine frequency vector and a sufficiently small analytic sampling function, then

1. *(existence) there exists a global solution $u(x,t)$;*
2. *(uniqueness) if $\tilde{u}$ is another solution on $\mathbb{R} \times [-T, T]$, and*

$$\tilde{u}, \partial_x^3 \tilde{u} \in L^\infty(\mathbb{R} \times [-T, T]),$$

   then $\tilde{u} = u$;

3. *(x-dependence) for each t, $u(\cdot, t)$ is quasi-periodic in x,*

$$u(x,t) = \sum_{m \in \mathbb{Z}^\nu} c(m,t) e^{2\pi i m\theta}$$

$$|c(m,t)| \leq \sqrt{4\varepsilon}\, e^{-\frac{\kappa_0}{4}|m|}$$

4. *(t-dependence) $t \mapsto u(\cdot, t)$ is $W^{k,\infty}(\mathbb{R})$-almost periodic in t, for any integer $k \geq 0$.*

Reflectionless operators and Remling's theorem

- Define Green's function of $H_W = -\partial_x^2 + W$ by

$$G(x, y; z) = \langle \delta_x, (H_W - z)^{-1} \delta_y \rangle$$

- W is reflectionless if

$$\operatorname{Re} G(0, 0; E + i0) = 0 \quad \text{for Lebesgue-a.e. } E \in S = \sigma(H_W)$$

  Write $W \in \mathcal{R}(S)$ in this case

Theorem (Remling 2007)

Assume W is almost periodic and $S = \sigma(H_W) = \sigma_{\mathrm{ac}}(H_W)$. Then $W \in \mathcal{R}(S)$.

Theorem (Rybkin 2008)

Assume that $V \in \mathcal{R}(S)$ and $\sigma_{\mathrm{ac}}(H_V) = S$. Assume that $u(x, t)$ is a solution such that

$$u, \partial_x^3 u \in L^\infty(\mathbb{R} \times [-T, T])$$

for some $T > 0$. Then, $u(\cdot, t) \in \mathcal{R}(S)$ for every $t \in [-T, T]$.

Torus of Dirichlet data

- Write the spectrum as $S = [\underline{E}, \infty) \setminus \bigcup_{j \in J} (E_j^-, E_j^+)$
- Fix a gap (E_j^-, E_j^+) and $x \in \mathbb{R}$
- Define $\mu_j(x) = \begin{cases} E & G(x,x;E) = 0, \text{where } E \in (E_j^-, E_j^+) \\ E_j^- & G(x,x;E) > 0, \forall E \in (E_j^-, E_j^+) \\ E_j^+ & G(x,x;E) < 0, \forall E \in (E_j^-, E_j^+) \end{cases}$
- If $\mu_j(x) \in (E_j^-, E_j^+)$, define $\sigma_j(x) \in \{\pm\}$, so that $\mu_j(x)$ is a Dirichlet eigenvalue of H on $[x, \sigma_j(x)\infty)$
- View $(\mu_j(x), \sigma_j(x))_{j \in J}$ as an element of a torus $\mathcal{D}(S) = \prod_{j \in J} \mathbb{T}_j$
- Introduce angular variables $\varphi_j(x) \in \mathbb{R}/2\pi\mathbb{Z}$ by

$$\mu_j = E_j^- + (E_j^+ - E_j^-)\cos^2(\varphi_j/2)$$
$$\sigma_j = \operatorname{sgn} \sin \varphi_j$$

The Dubrovin flow and the trace formula

Theorem (Craig 1989)

Under suitable conditions on S, the $\varphi_j(x)$ evolve according to the Dubrovin flow

$$\frac{d}{dx}\varphi(x) = \Psi(\varphi(x))$$

which is given by a Lipshitz vector field Ψ,

$$\Psi_j(\varphi) = \sigma_j \sqrt{4(\underline{E} - \mu_j)(E_j^+ - \mu_j)(E_j^- - \mu_j) \prod_{k \neq j} \frac{(E_k^- - \mu_j)(E_k^+ - \mu_j)}{(\mu_k - \mu_j)^2}},$$

and the trace formula recovers the potential,

$$V(x) = Q_1(\varphi(x)) := \underline{E} + \sum_{j \in J} (E_j^+ + E_j^- - 2\mu_j(x)).$$

KdV evolution on Dirichlet data

Add time dependence: consider a solution $u(x,t)$ and its Dirichlet data $\mu(x,t)$.

Proposition

Under suitable "Craig-type" conditions on S*,*

$$\partial_x \varphi(x,t) = \Psi(\varphi(x,t)), \qquad \partial_t \varphi(x,t) = \Xi(\varphi(x,t)),$$

where Ξ *is a Lipshitz vector field given by*

$$\Xi_j = -2(Q_1 + 2\mu_j)\Psi_j,$$

and the trace formula recovers the solution,

$$u(x,t) = Q_1(\varphi(x,t)) = \underline{E} + \sum_{j \in J} (E_j^+ + E_j^- - 2\mu_j(x,t)).$$

Existence of solutions

Under the Craig-type conditions on S, we prove

Proposition

Let $f \in \mathcal{D}(S)$. There exists $\varphi : \mathbb{R}^2 \to \mathcal{D}(S)$ such that $\varphi(0,0) = f$ and

$$\partial_x \varphi(x,t) = \Psi(\varphi(x,t)), \qquad \partial_t \varphi(x,t) = \Xi(\varphi(x,t)).$$

If we define $u : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(x,t) = Q_1(\varphi(x,t))$$

then the function $u(x,t)$ obeys the KdV equation. Moreover, for each $t \in \mathbb{R}$, we have $u(\cdot,t) \in \mathcal{R}(S)$ and $B(u(\cdot,t)) = \varphi(0,t)$.
Moreover, if we define $Q_k = \underline{E}^k + \sum_{j\in J}((E_j^-)^k + (E_j^+)^k - 2\mu_j^k)$, then

$$Q_2 \circ \varphi = -\tfrac{1}{2}\partial_x^2 u + u^2$$

$$Q_3 \circ \varphi = \frac{3}{16}\partial_x^4 u - \frac{3}{2} u \partial_x^2 u - \frac{15}{16}(\partial_x u)^2 + u^3$$

Proof is by showing convergence of approximants with finite gap spectra $S^N = [\underline{E}, \infty) \setminus \bigcup_{j=1}^N (E_j^-, E_j^+)$, for which the above statements were known.

Almost periodicity of the solution

Define $\xi_j(z)$ as the solution of the Dirichlet problem on $\mathbb{C} \setminus S$ with boundary values on $\bar{S}$ given by

$$\xi_j(x) = \begin{cases} 1 & x = \infty \text{ or } x \in S, x \geq E_j^+ \\ 0 & x \in S, x \leq E_j^- \end{cases}$$

Sodin–Yuditskii define the infinite dimensional Abel map $A : \mathcal{D}(S) \to \mathbb{T}^J$,

$$A_j(\varphi) = \pi \sum_{k \in J} \sigma_k \left(\xi_j(\mu_k) - \xi_j(E_k^-) \right) \pmod{2\pi\mathbb{Z}}$$

Proposition

The map A linearizes the KdV flow: for some $\delta, \zeta \in \mathbb{R}^J$,

$$A(\varphi(x,t)) = A(\varphi(0,0)) + \delta x + \zeta t.$$

The proof uses finite gap approximants, for which linearity is known,

$$A_j^N(\varphi^N(x,t)) = A_j^N(\varphi^N(0,0)) + \delta_j^N x + \zeta_j^N t,$$

and uniform convergence on compacts.

Thank you!