Eigensystem multiscale analysis for Anderson localization in energy intervals I

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joint work with Alexander Elgart

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Basic notation

- H will always denote a discrete Schrödinger operator, that is, an operator $H = -\Delta + V$ on $\ell^2(\mathbb{Z}^d)$, where where Δ is the (centered) discrete Laplacian: $(\Delta\varphi)(x) := \sum_{|y-x|=1} \varphi(y)$ for $\varphi \in \ell^2(\mathbb{Z}^d)$.

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$$\Lambda_L(x) = \Lambda_L^{\mathbb{R}}(x) \cap \mathbb{Z}^d, \text{ where } x \in \mathbb{R}^d \text{ and } \Lambda_L^{\mathbb{R}}(x) = \left\{ y \in \mathbb{R}^d;\ \|y - x\| \le \tfrac{L}{2} \right\}.$$

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Note that $(L-2)^d < |\Lambda_L(x)| \leq (L+1)^d$ for $L \geq 2$.

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- We call (φ, λ) an eigenpair for H_Θ if λ is an eigenvalue for H_Θ and φ is a corresponding normalized eigenfunction, that is,

$$H_\Theta \varphi = \lambda \varphi, \quad \text{where} \quad \lambda \in \mathbb{R} \text{ and } \varphi \in \ell^2(\Theta) \text{ with } \|\varphi\| = 1.$$

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- A collection $\{(\varphi_j, \lambda_j)\}_{j \in J}$ of eigenpairs for H_Θ will be called an eigensystem for H_Θ if $\{\varphi_j\}_{j \in J}$ is an orthonormal basis for $\ell^2(\Theta)$.

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- If Θ is finite and all eigenvalues of H_Θ are simple, we can rewrite an eigensystem as $\{(\varphi_\lambda, \lambda)\}_{\lambda \in \sigma(H_\Theta)}$.

Level spacing boxes and localized eigenfunctions

Definition

Given $L > 0$, a finite set $\Theta \subset \mathbb{Z}^d$ will be called L-level spacing for H if all eigenvalues of H_Θ are simple, and

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Let Λ_L be a box, $x \in \Lambda_L$, and $m \geq 0$. Then $\varphi \in \ell^2(\Lambda_L)$ is said to be (x,m)-localized if $\|\varphi\| = 1$ and

$$|\varphi(y)| \leq \mathrm{e}^{-m\|y-x\|} \quad \text{for all} \quad y \in \Lambda_L \quad \text{with} \quad \|y-x\| \geq L^\tau.$$

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Definition

Let $J = (E - B, E + B) \subset I = (E - A, E + A)$, where $E \in \mathbb{R}$ and $0 < B \leq A$, and let $m > 0$.

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$$h_I(t) = h\left(\tfrac{t-E}{A}\right) \quad \text{for} \quad t \in \mathbb{R}, \quad \text{where} \quad h(s) = \begin{cases} 1 - s^2 & \text{if } s \in [0,1) \\ 0 & \text{otherwise} \end{cases}.$$

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Note that $\chi_J(\nu)h_I(\nu)) > 0 \iff \nu \in J$.

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Remark: $\sigma(H_\omega) = \Sigma := [-2d, 2d] + \operatorname{supp}\mu$ with probability one.

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In the special case of a box Λ_L, we have

$$\mathbb{P}\{\Lambda_L \text{ is level spacing for } H_\omega\} \geq 1 - Y_\mu \left(L+1\right)^{2d} \mathrm{e}^{-(2\alpha-1)L^\beta}.$$

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Note that

$$\lim_{L\to\infty} A_\infty(A, L) = A \quad \text{and} \quad \lim_{L\to\infty} m_\infty(m, L, C) = m.$$

Eigensystem multiscale analysis in an energy interval

Theorem

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where $I_0 = (E - A_0, E + A_0) \subset \mathbb{R}$, with $E \in \mathbb{R}$ and $A_0 > 0$, and

$$m_- L_0^{-\kappa'} \leq m_0 \leq \tfrac{1}{2} \log\left(1 + \tfrac{A_0}{4d}\right).$$

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where $I_0 = (E - A_0, E + A_0) \subset \mathbb{R}$, with $E \in \mathbb{R}$ and $A_0 > 0$, and

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Let $A_\infty = A_\infty(A_0, L_0)$, $I_\infty = I_\infty(A_0, L_0)$, and $m_\infty = m_\infty(m_0, L_0, C_{d,m_-})$.

Eigensystem multiscale analysis in an energy interval

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Let $A_\infty = A_\infty(A_0, L_0)$, $I_\infty = I_\infty(A_0, L_0)$, and $m_\infty = m_\infty(m_0, L_0, C_{d,m_-})$. Then for all $L \geq L_0^\gamma$ we have

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- The usual forms of localization in an energy interval are commonly proved by either a Green's function multiscale analysis or the fractional moment method.

Localization on the energy interval

We fix $\nu > \frac{d}{2}$, and let T_0 be the operator on $\ell^2(\mathbb{Z}^d)$ given by multiplication by the function $T_0(x) := \langle x \rangle^{\nu}$, where $\langle x \rangle = \sqrt{1 + \|x\|^2}$.

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1. *H_ω has pure point spectrum in the interval I.*
2. *If ψ_λ is an eigenfunction of H_ω with eigenvalue $\lambda \in I$, then ψ_λ is exponentially localized with rate of decay $\frac{m}{20} h_I(\lambda)$, more precisely,*

$$|\psi_\lambda(x)| \le C_{\omega,\lambda} \left\| T_0^{-1} \psi \right\| e^{-\frac{m}{20} h_I(\lambda) \|x\|} \qquad \text{for all} \quad x \in \mathbb{R}^d.$$

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In particular, for all intervals $J_\zeta(L) = [E_0, E_0 + C_{d,\mu} L^{-\frac{2\zeta}{d}})$ and all $m > 0$,

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The conclusions of the Theorem and the Corollary hold in the interval $J_{\zeta,\infty}$. Note $\lim_{L_0\to\infty} A_{\zeta,\infty} L_0^{\frac{2\zeta}{d}} = C_{d,\mu}$ and $\lim_{L_0\to\infty} m_{\zeta,\infty} L^{\frac{2\zeta}{d}} = \frac{C_{d,\mu}}{9d}$.

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It follows that, given $\zeta \in (0,1)$, for $g \geq g_\zeta(L)$ and all $m > 0$ we have

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In particular, the conclusions of the Theorem and Corollary hold in the interval J_∞. Moreover, $\lim_{L_0\to\infty} A_\infty(L_0) = B$ and $\lim_{L_0\to\infty} m_\infty(L_0) = m$.

Decay of Green's functions in (m, I)-localizing boxes

Lemma

Fix $m_{-} > 0$. *Let* $I = (E - A, E + A)$, *with* $E \in \mathbb{R}$ *and* $A > 0$, *and* $m > 0$. *Suppose that* Λ_L *is* (m, I)*-localizing for* H, *where*

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Lemma

Fix $m_- > 0$. *Let* $I = (E - A, E + A)$, *with* $E \in \mathbb{R}$ *and* $A > 0$, *and* $m > 0$. *Suppose that* Λ_L *is* (m, I)*-localizing for* H, *where*

$$m_- L^{-\kappa'} \leq m \leq \tfrac{1}{2} \log\left(1 + \tfrac{A}{4d}\right).$$

Let $\lambda \in I_L$ *with* $\operatorname{dist}\{\lambda, \sigma(H_{\Lambda_L})\} \geq \mathrm{e}^{-L^\beta}$.

Then, letting $G_{\Lambda_L}(\lambda) = (H_{\Lambda_L} - \lambda)^{-1}$, *we have*

$$|G_{\Lambda_L}(\lambda; x, y)| \leq e^{-m'' h_I(\lambda) \|x-y\|} \quad \textit{for all} \quad x, y \in \Lambda_L \ \textit{with} \ \|x - y\| \geq \tfrac{L}{100},$$

where

$$m'' \geq m\left(1 - C_{d,m_-} L^{-(1-\tau)}\right).$$

The main difficulty

Let Λ_L be (m, l)-localizing for H, and let $\{(\varphi_\nu, \nu)\}_{\nu \in \sigma(H_{\Lambda_L})}$ be an (m, l)-localized eigensystem for H_{Λ_L}.

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$$|\varphi_\nu(x)\varphi_\nu(y)| \leq \mathrm{e}^{-m' h_I(\nu)\|x-y\|}, \quad \text{where} \quad m' \geq m(1 - 100 L^{\tau - 1}).$$

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$$G_{\Lambda_L}(\lambda; x, y) = \sum_{\nu \in \sigma_I(H_{\Lambda_\ell})} (\nu - \lambda)^{-1} \overline{\varphi_\nu(x)} \varphi_\nu(y) + \sum_{\nu \in \sigma_{\mathbb{R} \setminus I}(H_{\Lambda_\ell})} (\nu - \lambda)^{-1} \overline{\varphi_\nu(x)} \varphi_\nu(y).$$

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We know $$\left| \sum_{\nu \in \sigma_I(H_{\Lambda_\ell})} (\nu - \lambda)^{-1} \overline{\varphi_\nu(x)} \varphi_\nu(y) \right| \leq \mathrm{e}^{L^\beta} \sum_{\nu \in \sigma_I(H_{\Lambda_\ell})} \mathrm{e}^{-m' h_I(\nu)\|x-y\|}.$$

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How can we estimate $\sum_{\nu\in\sigma_{\mathbb{R}\setminus I}(H_{\Lambda_\ell})}(\nu-\lambda)^{-1}\overline{\varphi_\nu(x)}\varphi_\nu(y)$? We have no information on φ_ν for $\nu\notin I$. Where does the decay comes from?