Adiabatic theorems in quantum statistical mechanics and Landauer principle

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ADIABATIC THEOREMS IN QSM

- Hilbert space $\mathcal{H}$, $\dim \mathcal{H} < \infty$, $H(t) = H + V(t)$, $t \in [0, 1]$, $V(0) = 0$.

$$\rho_{\mathrm{i}} = \mathrm{e}^{-\beta H(0)}/Z, \quad \rho_{\mathrm{f}} = \mathrm{e}^{-\beta H(1)}/Z.$$

- $T > 0$ adiabatic parameter, $U_T(t)$ time-evolution generated by $H(t/T)$ over the time interval $[0, T]$.

$$\rho_{\mathrm{i}}(T) = U_T^*(T)\rho_{\mathrm{i}} U_T(T).$$

- Before taking the adiabatic limit $T \to \infty$ we need to take first the TD (thermodynamic) limit. The limiting objects are denoted by the superscript (∞).

- Adiabatic theorem for thermal states (Araki-Avron-Elgart):

$$\lim_{T\to\infty} \|\rho_{\mathsf{i}}^{(\infty)}(T) - \rho_{\mathsf{f}}^{(\infty)}\| = 0.$$

Proof: Combination of the Avron-Elgart gapless adiabatic theorem and Araki's theory of perturbation of KMS structure.
Assumption: Ergodicity of TD limit quantum dynamical system w.r.t. instantaneous dynamics.

- Adiabatic theorem for relative entropy:

$$\lim_{T\to\infty} S(\rho_{\mathsf{i}}^{(\infty)}(T)|\rho_{\mathsf{i/f}}^{(\infty)}) = S(\rho_{\mathsf{f}}^{(\infty)}|\rho_{\mathsf{i/f}}^{(\infty)}).$$

$S(A|B) = \mathrm{tr}(A(\log A - \log B))$.

- Adiabatic theorem for Renyi's relative entropy:

$$\lim_{T\to\infty} S_{\mathrm{i}\alpha}(\rho_{\mathrm{i}}^{(\infty)}(T)|\rho_{\mathrm{i}}^{(\infty)}) = S_{\mathrm{i}\alpha}(\rho_{\mathrm{f}}^{(\infty)}|\rho_{\mathrm{i}}^{(\infty)}).$$

$$S_{\mathrm{i}\alpha}(A|B) = \mathrm{tr}(A^{1-\mathrm{i}\alpha}B^{\mathrm{i}\alpha}).$$

- Adiabatic theorem for FCS. Let $\mathbb{P}_T^{(\infty)}$ be the probability measure on $\mathbb{R}$ describing the statistics of energy differences ΔE in two times measurement protocol of the total energy (initially and at the time T).

$$\lim_{T\to\infty} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\alpha\Delta E} \mathrm{d}\mathbb{P}_T^{(\infty)}(\Delta E) = S_{-\mathrm{i}\alpha/\beta}.$$

LANDAUER PRINCIPLE

- Finite level quantum system $\mathcal{S}$ coupled to a thermal reservoir $\mathcal{R}$ ($\mathcal{H}_{\mathcal{R}}$, $H_{\mathcal{R}}$). $\dim \mathcal{H}_{\mathcal{S}} = d$, $\rho_{\mathcal{S},\mathsf{i}} = \mathbb{I}/d$, $\rho_{\mathcal{S},\mathsf{f}} > 0$ the final (target state). Landauer principle concerns energetic cost of the state transition $\rho_{\mathcal{S},\mathsf{i}} \to \rho_{\mathcal{S},\mathsf{f}}$ mediated by $\mathcal{R}$.

- Coupled system: $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}$, $H = H_{\mathcal{R}}$, $V(t)$ local interaction, $V(0) = 0$,

$$V(1) = -\frac{1}{\beta} \log \rho_{\mathcal{S},\mathsf{f}},$$

$$H(t) = H_{\mathcal{R}} + V(t),$$

$$\rho_{\mathsf{i}/\mathsf{f}} = \mathrm{e}^{-\beta H(0/1)}/Z = \rho_{\mathcal{S},\mathsf{i}/\mathsf{f}} \otimes \mathrm{e}^{-\beta H_{\mathcal{R}}}/Z.$$

- First TD limit, then adiabatic limit. The transition $\rho_{\mathcal{S},\mathrm{i}} \to \rho_{\mathcal{S},\mathrm{f}}$ follows from $\lim_{T\to\infty} \|\rho_{\mathrm{i}}^{(\infty)}(T) - \rho_{\mathrm{f}}^{(\infty)}\| = 0$.

- Landauer bound: The balance equation

$$\Delta S_T + \sigma_T = \beta \Delta Q_T$$

where, with $S(\sigma) = -\mathrm{tr}(\sigma \log \sigma)$,

$$\Delta S_T = S(\rho_{\mathcal{S},\mathrm{i}}(T)) - S(\rho_{\mathcal{S},\mathrm{i}}),$$

$$\Delta Q_T = \mathrm{tr}(\rho_{\mathrm{i}}(T) H_{\mathcal{R}}) - \mathrm{tr}(\rho_{\mathrm{i}} H_{\mathcal{R}}),$$

$$\sigma_T = S(\rho_{\mathrm{i}}(T) | \rho_{\mathcal{S},\mathrm{i}}(T) \otimes \mathrm{e}^{-\beta H_{\mathcal{R}}}/Z).$$

$\sigma_T \geq 0$ is the entropy production term, and the Landauer bound follows

$$\Delta S_T \geq \beta \Delta Q_T.$$

- After the TD limit, the adiabatic theorem for relative entropy

$$\lim_{T\to\infty} S(\rho_{\mathsf{i}}^{(\infty)}(T)|\rho_{\mathsf{i}}^{(\infty)}) = S(\rho_{\mathsf{f}}^{(\infty)}|\rho_{\mathsf{i}}^{(\infty)})$$

gives the saturation of the Landauer bound in the adiabatic limit: $\lim_{T\to\infty} \sigma_T = 0$,

$$S(\rho_{\mathcal{S},\mathsf{i}}) - S(\rho_{\mathcal{S},\mathsf{f}}) = \lim_{T\to\infty} \Delta S_T^{(\infty)} = \lim_{T\to\infty} \beta\Delta Q_T^{(\infty)}.$$

- Additional limit $\rho_{\mathcal{S},\mathsf{f}} \to |\psi\rangle\langle\psi|$ gives the familiar form

$$\log d = \beta\Delta\bar{Q}^{(\infty)}.$$

- Full Counting Statistics goes beyond mean values and captures fluctuations.

- The adiabatic theorem for FCS gives

$$\lim_{T\to\infty}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\alpha\Delta E}\mathrm{d}\mathbb{P}_T^{(\infty)} = S_{-\mathrm{i}\alpha/\beta} = \mathrm{tr}\left(\rho_{\mathrm{f}}\mathrm{e}^{\mathrm{i}\frac{\alpha}{\beta}(\log d+\log\rho_{\mathrm{f}})}\right)$$

- If $\rho_{\mathcal{S},\mathrm{f}} = \sum p_k|k\rangle\langle k|$, then $\lim_{T\to\infty}\mathbb{P}_T^{(\infty)} = \bar{\mathbb{P}}^{(\infty)}$, where

$$\bar{\mathbb{P}}^{\infty}\left(\frac{1}{\beta}(\log d + \log p_k)\right) = p_k.$$

The heat is a discrete random variable, and each allowed quanta of heat corresponds to a transition to a certain level of the final state.

- The atomic measure $\bar{\mathbb{P}}^{(\infty)}$ describes the heat fluctuations around the mean value given by the Landauer bound

$$\int_{\mathbb{R}} \Delta E \mathrm{d}\bar{\mathbb{P}}^{(\infty)}(\Delta E) = S(\rho_{\mathcal{S},\mathrm{i}}) - S(\rho_{\mathcal{S},\mathrm{f}}).$$

- In the limit $\rho_{\mathcal{S},\mathrm{f}} \to |\psi\rangle\langle\psi|$, $\bar{\mathbb{P}}^{(\infty)} \to \delta_{\beta^{-1}\log d}$, together with convergence of all momenta.

- At the same time

$$\lim_{\rho_{\mathcal{S},\mathrm{f}} \to |\psi\rangle\langle\psi|} \int_{\mathbb{R}} \mathrm{e}^{\alpha \Delta E} \mathrm{d}\bar{\mathbb{P}}^{(\infty)} = \begin{cases} e^{\frac{\alpha}{\beta}\log d} & \text{if} \quad \alpha > -\beta, \\ 1 & \text{if} \quad \alpha = -\beta, \\ \infty & \text{if} \quad \alpha < -\beta. \end{cases}$$

- We expect that this divergence is experimentally observable via recently proposed interferometry and control protocols for measuring FCS using an ancilla coupled to the joint system $\mathcal{S} + \mathcal{R}$.