Tsirelson’s problem and linear system games

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includes joint work with Richard Cleve and Li Liu
Non-local games

Win/lose based on outputs $a, b$ and inputs $x, y$

Alice and Bob must cooperate to win

Winning conditions known in advance
Non-local games

Win/lose based on outputs $a, b$ and inputs $x, y$

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Winning conditions known in advance

Complication: players cannot communicate while the game is in progress
Suppose game is played many times, with inputs drawn from some public distribution $\pi$.

To outside observer, Alice and Bob’s strategy is described by:

$$P(a, b|x, y) = \text{the probability of output } (a, b) \text{ on input } (x, y)$$

**Correlation matrix**: collection of numbers $\{P(a, b|x, y)\}$
Classical and quantum strategies

\[ P(a, b|x, y) = \text{the probability of output } (a, b) \text{ on input } (x, y) \]

Value of game \( \omega = \text{winning probability using strategy } \{P(a, b|x, y)\} \)

What type of strategies might Alice and Bob use?

Classical: can use randomness, flip coin to determine output.

Correlation matrix will be \( P(a, b|x, y) = A(a|x)B(b|y) \).

Quantum: Alice and Bob can share entangled quantum state

Bell’s theorem: Alice and Bob can do better with an entangled quantum state than they can do classically

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Quantum strategies

How do we describe a quantum strategy?

Use axioms of quantum mechanics:

- Physical system described by (finite-dimensional) Hilbert space
- No communication $\implies$ Alice and Bob each have their own (finite dimensional) Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$
- Hilbert space for composite system is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- Shared quantum state is a unit vector $|\psi\rangle \in \mathcal{H}$
- Alice’s output on input $x$ is modelled by measurement operators $\{M^x_a\}_a$ on $\mathcal{H}_A$
- Similarly Bob has measurement operators $\{N^y_b\}_b$ on $\mathcal{H}_B$

Quantum correlation: $P(a, b|x, y) = \langle \psi | M^x_a \otimes N^y_b | \psi \rangle$
Quantum correlations

Set of quantum correlations:

\[ C_q = \left\{ P(a, b| x, y) \right\} : P(a, b| x, y) = \langle \psi | M^x_a \otimes N^y_b | \psi \rangle \text{ where} \]
\[ |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \text{ where } \mathcal{H}_A, \mathcal{H}_B \text{ fin dim'} \]
\[ M^x_a \text{ and } N^y_b \text{ are projections on } \mathcal{H}_A \text{ and } \mathcal{H}_B \]
\[ \sum_a M^x_a = I \text{ and } \sum_b N^y_b = I \text{ for all } x, y \]

Two variants:

1. \( C_{qs} \): Allow \( \mathcal{H}_A \) and \( \mathcal{H}_B \) to be infinite-dimensional

2. \( C_{qa} = \overline{C_q} \): limits of finite-dimensional strategies

Relations: \( C_q \subseteq C_{qs} \subseteq C_{qa} \)
Commuting-operator model

Another model for composite systems: *commuting-operator model*

In this model:

- Alice and Bob each have an algebra of observables $\mathcal{A}$ and $\mathcal{B}$
- $\mathcal{A}$ and $\mathcal{B}$ act on the joint Hilbert space $\mathcal{H}$
- $\mathcal{A}$ and $\mathcal{B}$ commute: if $a \in \mathcal{A}$, $b \in \mathcal{B}$, then $ab = ba$.

This model is used in quantum field theory

Correlation set:

$$C_{qc} := \left\{ \{P(a, b|x, y)\} : P(a, b|x, y) = \langle \psi | M_a^x N_b^y | \psi \rangle, \quad M_a^x N_b^y = N_b^y M_a^x \right\}$$

Hierarchy: $C_q \subseteq C_{qs} \subseteq C_{qa} \subseteq C_{qc}$
Tsirelson’s problem

Two models of QM: tensor product and commuting-operator

Tsirelson problems: is $C_t, t \in \{q, qs, qa\}$ equal to $C_{qc}$

Fundamental questions:

1. What is the power of these models?
   
   **Strong Tsirelson**: is $C_q = C_{qc}$?

2. Are there observable differences between these two models, accounting for noise and experimental error?
   
   **Weak Tsirelson**: is $C_{qa} = C_{qc}$?
What do we know?

Theorem (Ozawa, JNPPSW, Fr)

\[ C_{qa} = C_{qc} \text{ if and only if Connes’ embedding problem is true} \]

Theorem (S)

\[ C_{qs} \neq C_{qc} \]
### Other fundamental questions

**Question:** Given a non-local game, can we compute the optimal value $\omega_t$ over strategies in $C_t$, $t \in \{qa, qc\}$?

<table>
<thead>
<tr>
<th>Theorem (Navascués, Pironio, Acín)</th>
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<tbody>
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<td><em>Given a non-local game, there is a hierarchy of SDPs which converge in value to $\omega_{qc}$</em></td>
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**Problem:** no way to tell how close we are to the correct answer

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<th>Theorem (S)</th>
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<td><em>It is undecidable to tell if $\omega_{qc} &lt; 1$</em></td>
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Two theorems

Theorem (S)

\[ C_{qs} \neq C_{qc} \]

Theorem (S)

*It is undecidable to tell if* \( \omega_{qc} < 1 \)

Proofs: make connection to group theory via linear system games
Linear system games

Start with \( m \times n \) linear system \( Ax = b \) over \( \mathbb{Z}_2 \)

\[ \implies \text{Get a non-local game } G, \text{ and } \]

\[ \implies \text{a solution group } \Gamma \]

\( \Gamma \): Group generated by \( X_1, \ldots, X_n \), satisfying relations

1. \( X_j^2 = [X_j, J] = J^2 = e \) for all \( j \)

2. \( \prod_{j=1}^{n} X_j^{A_{ij}} = J^{b_i} \) for all \( i \)

3. If \( A_{ij}, A_{ik} \neq 0 \), then \( [X_j, X_k] = e \).
Quantum solutions of $Ax = b$

Solution group $\Gamma$: Group generated by $X_1, \ldots, X_n$, satisfying relations

1. $X_j^2 = [X_j, J] = J^2 = e$ for all $j$
2. $\prod_{j=1}^n X_j^{A_{ij}} = J^{b_i}$ for all $i$
3. If $A_{ij}, A_{ik} \neq 0$, then $[X_j, X_k] = e$.

Theorem (Cleve-Mittal, Cleve-Liu-S)

Let $G$ be the game for linear system $Ax = b$. Then:

- $G$ has a perfect strategy in $C_{qs}$ if and only if $\Gamma$ has a finite-dimensional representation with $J \neq I$
- $G$ has a perfect strategy in $C_{qc}$ if and only if $J \neq e$ in $\Gamma$
**Group embedding theorem**

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- $G$ has a perfect strategy in $C_{qc}$ if and only if $J \neq e$ in $\Gamma$

**Theorem (S)**

Let $G$ be any finitely-presented group, and suppose we are given $J_0$ in the center of $G$ such that $J_0^2 = e$.

Then there is an injective homomorphism $\phi : G \hookrightarrow \Gamma$, where $\Gamma$ is the solution group of a linear system $Ax = b$, with $\phi(J_0) = J$. 
How do we prove the embedding theorem?

**Theorem (S)**

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Given finitely-presented group $G$, we get $\Gamma$ from a linear system

But what linear system?

Linear systems over $\mathbb{Z}_2$ correspond to vertex-labelled hypergraphs

So we can answer this pictorially by writing down a hypergraph...
The hypergraph by example

\[ \langle x, y, z, u, v : xyxz = xuvu = e = x^2 = y^2 = \cdots = v^2 \rangle \]

does not include preprocessing
\[ \langle x, y, z, u, v : xyxz = xuvu = e = x^2 = y^2 = \cdots = v^2 \rangle \]

Thank-you!