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Categories and Quantum Computing

Carlos M. Ortiz Marrero
Pacific Northwest National Laboratory

Joint work with Paul Bruillard

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Overview

1. Topological Quantum Computing

2. Categories

3. Classification by Rank

Topological Quantum Computing

What is Quantum Computation?

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Definition [Freedman, Kitaev, Larsen, Wang '03]

Quantum Computation is any computational model based upon the theoretical ability to manufacture, manipulate, and measure quantum states

What is Topological Quantum Computation?

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Definition [Freedman, Kitaev, Larsen, Wang '03]

Topological Quantum Computation is any computational model based upon the theoretical ability to manufacture, manipulate, and measure quantum states with topological phases of matter.

Topological Quantum Field Theories

Definition [Nayak, et al ’08]

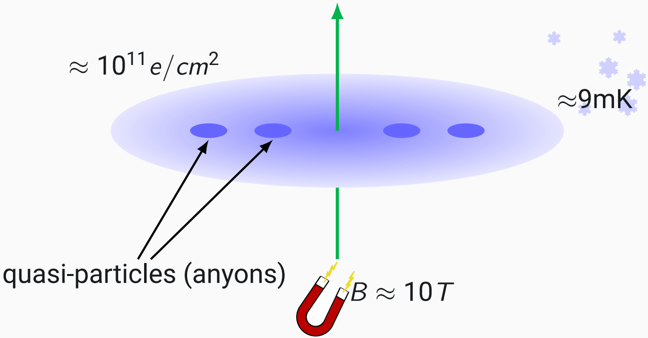
A topological phase of a matter (TPM) is a physical system such that its low-energy effective field theory* is described by a TQFT.

Definition [Witten, et al ‘88]

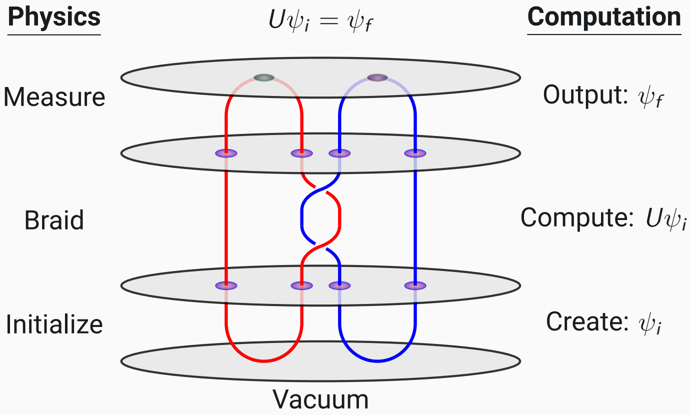
A topological quantum field theory (TQFT) is quantum mechanical model where “amplitudes only depend on the topology of the process”.

*“...system is away from any boundary and has low energy and temperature.”

Example: Two-Dimensional Electron Gas



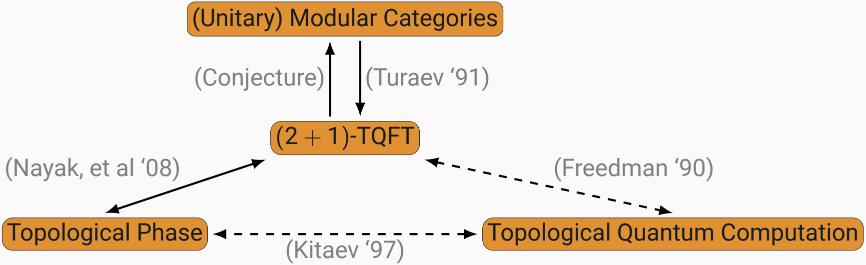
- These things exist! (e.g. $GaAs$, α-$RuCl_3$)
- There is theoretical (and some experimental) evidence that you can perform quantum computation with some of these phases.
- Nobel prizes: experimental (1985, 1998) and theoretical (2016).



- Gates are given by unitary representations of the braid group.
- Computation is topologically protected from decoherence.

Mathematical Structure

The appropriate mathematical structure is a modular category.



Morally, a classification of modular categories gives you a classification of topological phases.

Categories

Representation Category of a Group

$$\mathrm{Rep}(G)$$

Basic properties:

- $(\mathrm{Rep}(G), \oplus, \otimes, *)$
- $Hom^G(\rho, \varphi)$ is a finite dimentional vector space
- $|Irr(G)| < \infty$
- $\phi = \bigoplus_k \alpha_k \psi_k, \psi_k \in Irr(G)$

Definition

A Premodular category is a spherical, braided, fusion category.

- Abelian Monodial Category $(\mathcal{C}, \oplus, \otimes)$
- $\mathbb{C}$-linear: $Hom(X, Y)$ is a finite dimensional vector space
- finite rank: Finite number of simple classes $\{X_0 = \mathbb{1}, X_1, ..., X_n\}$
- semisimple: $X \cong \bigoplus_k \mu_k X_k$
- Dual object: X^* makes sense
- $X \otimes Y \cong Y \otimes X$
- $X^{**} \cong X$ and $Tr_{\mathcal{C}}$

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Key Diference: Elements of $\mathcal{C}$ have no internal structure.

Categorical Data

These set of axioms give rise to data that is an invariant for categories,

- $S = (s_{XY})$
- $\theta_X =$ root of unity **[Vafa '88]**

Definition

If $\mathcal{C}$ is premodular and $Det(S) \neq 0$, we say $\mathcal{C}$ is a modular category.

Mathematical Importance of Fusion Categories

We can think of the theory of fusion categories as an extension of representation theory:

Theorem [Deligne, Milne '82]

$\mathrm{Rep}(G)$, regarded as a fusion category, uniquely determines the group G up to isomorphism.

Mathematical Importance of Fusion Categories

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Remark

You get modular categories from von Neumann Algebras, vertex operator algebras, Hopf algebras, and Quantum Groups.

Physical meaning of Categorical data

Categorical Data	Anyonic System
$\mathbb{1}$	Vacuum state
X_i	Particle type
X_i^*	Antiparticle
θ_X	Particle statistics
$Det(S) \neq 0$	Particles are distinguishable
$Rank(S) = 1$	Particles exchange is boring

Definition

For $X \in Irr(C)$, we define $d_X := Tr_{\mathcal{C}}(Id_X)$ to be the quantum dimension of X.

Conjeture [Naidu, Rowell '11]

X gives rise to a universal gate set (via particle exchange) $\iff d_X^2 \notin \mathbb{Z}$

Classification by Rank

Classification by rank

Theorem [Bruillard, Ng, Rowell, Wang '13]

There are finitely many modular categories of a given rank r.

- Complete classification up to rank 5.

Conjeture

There are finitely many premodular categories of a given rank r.

- Complete clasification up to rank 4 **[Bruillard]**.

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Reference

On ArXiv:

P. Bruillard, *Rank 4 premodular categories*

P. Bruillard, C. Ortiz, *Rank 5 premodular categories* (coming soon...)

Thanks!