Discriminating quantum states: the *multiple Chernoff distance*

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Accessing quantum systems: quantum measurement

- Quantum measurement: formulated as positive operator-valued measure (POVM)

\[ \mathcal{M} = \{M_i\}_i, \text{ with } 0 \leq M_i \leq \mathbb{1} \text{ and } \sum_i M_i = \mathbb{1} ; \]

when performing the POVM on a system in the state \( \omega \), we obtain outcome "i" with probability

\[ \text{Tr}(\omega M_i). \]

- von Neumann measurement: special case of POVM, with the POVM elements being orthogonal projectors:

\[ M_i M_j = \delta_{ij} M_i, \text{ where } \delta_{ij} \text{ is the Kronecker delta.} \]
Quantum state discrimination (quantum hypothesis testing)

- Suppose a quantum system is in one of a set of states \( \{\omega_1, \ldots, \omega_r\} \), with a given prior \( \{p_1, \ldots, p_r\} \). The task is to detect the true state with a minimal error probability.

- Method: making quantum measurement \( \{M_i\}_{i=1}^r \).

- Error probability (let \( A_i := p_i \omega_i \))

\[
P_e (\{A_1, \ldots, A_r\}; \{M_1, \ldots, M_r\}) := \sum_{i=1}^r \text{Tr} A_i (1 - M_i).
\]

- Optimal error probability

\[
P_e^* (\{A_1, \ldots, A_r\}) := \min \left\{ P_e (\{A_1, \ldots, A_r\}; \{M_1, \ldots, M_r\}) : \text{POVM} \ \{M_1, \ldots, M_r\} \right\}.
\]
Asymptotics in quantum hypothesis testing

- What's the asymptotic behavior of 
  \( P_e^* \left( \{p_1 \rho_1^{\otimes n}, \ldots, p_r \rho_r^{\otimes n}\} \right) \), as \( n \to \infty \)?

- Exponentially decay! (Parthasarathy '2001)
  \[ P_e^* \sim \exp(-\xi n) \]

- But, what's the error exponent
  \[ \xi = \liminf_{n \to \infty} \frac{-1}{n} \log P_e^* \left( \{p_1 \rho_1^{\otimes n}, \ldots, p_r \rho_r^{\otimes n}\} \right) \]?

It has been an open problem (except for \( r=2 \)!)
Outline

1. The problem
2. The answer
3. History review
4. Proof sketch
5. One-shot case
6. Open questions
Our result:
error exponent = multiple Chernoff distance

We prove that

**Theorem**  Let \( \{\rho_1, \ldots, \rho_r\} \) be a finite set of quantum states on a finite-dimensional Hilbert space \( \mathcal{H} \). Then the asymptotic error exponent for testing \( \{\rho_1^\otimes n, \ldots, \rho_r^\otimes n\} \), for an arbitrary prior \( \{p_1, \ldots, p_r\} \), is given by the multiple quantum Chernoff distance:

\[
\lim_{n \to \infty} \frac{-1}{n} \log P_e^*(\{p_1 \rho_1^\otimes n, \ldots, p_r \rho_r^\otimes n\}) = \min_{(i,j):i \neq j} \max_{0 \leq s \leq 1} \left\{ -\log \text{Tr} \rho_i^s \rho_j^{1-s} \right\}.
\]
Remarks

Remark 1: Our result is a multiple-hypothesis generalization of the $r=2$ case. Denote the multiple quantum Chernoff distance (r.h.s. of eq. (1)) as $C(\rho_1, \ldots, \rho_r)$, then

$$C(\rho_1, \ldots, \rho_r) = \min_{(i,j):i\neq j} C(\rho_i, \rho_j),$$

with the binary quantum Chernoff distance is defined as

$$C(\rho_1, \rho_2) := \max_{0 \leq s \leq 1} \{ - \log \mathrm{Tr} \rho_1^s \rho_2^{1-s} \}.$$ 

Remark 2: when $\rho_1, \ldots, \rho_r$ commute, the problem reduces to classical statistical hypothesis testing. Compared to the classical case, the difficulty of quantum statistics comes from noncommutativity & entanglement.
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The classical Chernoff distance as the optimal error exponent for testing two probability distributions was given in H. Chernoff, Ann. Math. Statist. 23, 493 (1952).

The multiple generalizations were subsequently made in

N. P. Salihov, Dokl. Akad. Nauk SSSR 209, 54 (1973);
E. N. Torgersen, Ann. Statist. 9, 638 (1981);
Some history review

- Quantum hypothesis testing (state discrimination) was the main topic in the early days of quantum information theory in 1970s.

- Maximum likelihood estimation
  - for two states: Holevo-Helstrom tests
    \[
    \left( \{ \rho_1 - \rho_2 > 0 \}, 1 - \{ \rho_1 - \rho_2 > 0 \} \right)
    \]
  - for more than two states: only formulated in a complex and implicit way. **Competitions between pairs** make the problem complicated!
Some history review

- In 2001, Parthasarathy showed exponential decay.

- In 2006, two groups [Audenaert et al] and [Nussbaum & Szkola] together solved the r=2 case.

- In 2010/2011, Nussbaum & Szkola conjectured the solution (our theorem), and proved that $C/3 \leq \xi \leq C$.

- In 2014, Audenaert & Mosonyi proved that $C/2 \leq \xi \leq C$.
Sketch of proof

- We only need to prove the achievability part \( \xi \geq C \).
  
  For this purpose, we construct an *asymptotically optimal quantum measurement*, and show that it achieves the quantum multiple Chernoff distance as the error exponent.

- Motivation: consider detecting two weighted pure states.
  
  **Big overlap**: give up the light one;
  
  **Small overlap**: make a projective measurement, using orthonormalized version of the two states.
Sketch of proof

Spectral decomposition:

\[ \rho_i^{\otimes n} = \bigoplus_{k=1}^{T_i} \lambda_{ik}^{(n)} Q_{ik}^{(n)}, \]

\[ T := \max \{ T_i \} i \leq (n + 1)^d \]

Overlap between eigenspaces:

\[ \text{Olap} \left( \text{supp} \left( Q_{ik}^{(n)} \right), \text{supp} \left( Q_{j\ell}^{(n)} \right) \right) \]

\[ := \max \left\{ |\langle \varphi | \phi \rangle| : |\varphi \rangle \in \text{supp} \left( Q_{ik}^{(n)} \right), |\phi \rangle \in \text{supp} \left( Q_{j\ell}^{(n)} \right) \right\} \]
Sketch of proof

"Dig holes" in every eigenspaces to reduce overlaps

\[ \tilde{\rho}_1^{\otimes n}, \tilde{\rho}_2^{\otimes n}, \ldots, \tilde{\rho}_r^{\otimes n} \]

\[ \epsilon \text{-subtraction:} \]

Let \( P_1 P_2 P_1 = \bigoplus_x \lambda_x Q_x \)

Define \( P_1 \ominus_\epsilon P_2 := P_1 - \sum_{x: \lambda_x \geq \epsilon^2} Q_x \)

\[ \tilde{\rho}_i^{\otimes n} = \bigoplus_{k=1}^{T_i} \lambda_{ik}^{(n)} Q_{ik}^{(n)} , \quad \text{Olap} \left( \text{supp} \left( Q_{ik}^{(n)} \right), \text{supp} \left( Q_{j\ell}^{(n)} \right) \right) \leq \epsilon \]
Sketch of proof

- Now the supporting space of the hypothetic states have small overlaps. For $i \neq j$, $\text{Olap} \left( \text{supp} \left( \rho_i^{\otimes n} \right), \text{supp} \left( \rho_j^{\otimes n} \right) \right) \leq T\epsilon$

- The next step is to orthogonalize these eigenspaces
  1. Order the eigenspaces according to their eigenvalues, in the decreasing order.
  2. Orthogonalization using the Gram-Schmidt process.
Sketch of proof

- Now the eigenspaces are all orthogonal.
  \[ \rho_i^{\otimes n} = \bigoplus_{k=1}^T_i \lambda_{i_k}^{(n)} \tilde{Q}_{i_k}^{(n)} \]

- We construct a projective measurement
  \[ \left\{ \Pi_i = \bigoplus_k Q_{i_k}^{(n)} \right\}_{i=1}^r \]

- Use this to discriminate the original states:
  \[ P_{\text{succ}} = \sum_{i=1}^r p_i \text{Tr} \rho_i^{\otimes n} \Pi_i \]
Sketch of proof

\( Q_{ik}^{(n)} \) “digging holes” \( \tilde{Q}_{ik}^{(n)} \) orthogonalization \( \tilde{Q}_{ik}^{(n)} \)

- Loss in "digging holes":
  \[
  \text{Tr} \left( Q_{ik}^{(n)} - \tilde{Q}_{ik}^{(n)} \right) \leq \frac{1}{\epsilon^2} \sum_{(j,\ell) : \lambda_{j\ell}^{(n)} > \lambda_{ik}^{(n)}} \text{Tr} Q_{ik}^{(n)} Q_{j\ell}^{(n)}
  \]

- Mismatch due to orthogonalization:
  \[
  \text{Tr} \left[ \tilde{Q}_{ik}^{(n)} \left( \mathbb{1} - \tilde{Q}_{ik}^{(n)} \right) \right] \leq \frac{1 - (r - 1)T \epsilon}{1 - 2(r - 1)T \epsilon} \sum_{(j,\ell) : \lambda_{j\ell}^{(n)} > \lambda_{ik}^{(n)}} \text{Tr} Q_{ik}^{(n)} Q_{j\ell}^{(n)}
  \]

- Estimation of the total error:
  \[
  P_e \leq \sum_{(i,k)} \lambda_{ik}^{(n)} \text{Tr} \left[ Q_{ik}^{(n)} \left( \mathbb{1} - \tilde{Q}_{ik}^{(n)} \right) \right] \leq \sum_{(i,k)} \lambda_{ik}^{(n)} \left\{ \text{Tr} \left( Q_{ik}^{(n)} - \tilde{Q}_{ik}^{(n)} \right) + \text{Tr} \left[ \tilde{Q}_{ik}^{(n)} \left( \mathbb{1} - \tilde{Q}_{ik}^{(n)} \right) \right] \right\}
  \]
\[ P_e \leq \left( \frac{1}{\epsilon^2} + \frac{1 - (r - 1)T\epsilon}{1 - 2(r - 1)T\epsilon} \right) \sum_{(i,j):i \neq j} \sum_{k,\ell} \min\{\lambda_{ik}^{(n)}, \lambda_{j\ell}^{(n)}\} \text{Tr} Q_{ik}^{(n)} Q_{j\ell}^{(n)} \]

\[ \leq p(n) \]

\[ \leq p(n) \sum_{(i,j):i \neq j} \min_{0 \leq s \leq 1} \left( \text{Tr} \rho_i^s \rho_j^{(1-s)} \right)^n \]

\[ \sim \exp \left\{ -n \left( \min_{(i,j):i \neq j} \max_{0 \leq s \leq 1} \left\{ -\log \text{Tr} \rho_i^s \rho_j^{1-s} \right\} \right) \right\} \]
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Result for the one-shot case

**Theorem** Let $A_1, \ldots, A_r \in \mathcal{P}(\mathcal{H})$ be nonnegative matrices on a finite-dimensional Hilbert space $\mathcal{H}$. For all $1 \leq i \leq r$, let $A_i = \bigoplus_{k=1}^{T_i} \lambda_{ik} Q_{ik}$ be the spectral decomposition of $A_i$, and write $T := \max\{T_1, \ldots, T_r\}$. Then

$$P_e^*(\{A_1, \ldots, A_r\}) \leq 10(r-1)^2 T^2 \sum_{(i,j):i<j} \sum_{k,\ell} \min\{\lambda_{ik}, \lambda_{j\ell}\} \text{Tr} Q_{ik} Q_{j\ell}.$$ 

Remark 1: It matches a lower bound up to some states-dependent factors:

$$P_e^*(\{A_1, \ldots, A_r\}) \geq \frac{1}{2(r-1)} \sum_{(i,j):i<j} \sum_{k,\ell} \min\{\lambda_{ik}, \lambda_{j\ell}\} \text{Tr} Q_{ik} Q_{j\ell}.$$ 

Result for the one-shot case

- Remark 2: for the case $r=2$, we have

\[
P^{*}_{\epsilon} (\{A_1, A_2\}) \leq 10T^2 \sum_{k, \ell} \min\{\lambda_{1k}, \lambda_{2\ell}\} \text{Tr} \ Q_{1k}Q_{2\ell}.
\]

On the other hand, it is proved in [K. Audenaert et al, PRL, 2007] that

\[
P^{*}_{\epsilon} (\{A_1, A_2\}) \leq \min_{0 \leq s \leq 1} \text{Tr} \ A_1^s A_2^{1-s}.
\]

(note that it is always true that

\[
\sum_{k, \ell} \min\{\lambda_{1k}, \lambda_{2\ell}\} \text{Tr} \ Q_{1k}Q_{2\ell} \leq \min_{0 \leq s \leq 1} \text{Tr} \ A_1^s A_2^{1-s}.
\]
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Open questions

1. Applications of the bounds:

\[
P_e^* (\{A_1, \ldots, A_r\}) \begin{cases} 
\leq 10(r-1)^2T^2 \sum_{(i,j):i<j} \sum_{k,\ell} \min\{\lambda_{ik}, \lambda_{j\ell}\} \Tr Q_{ik}Q_{j\ell} \\
\geq \frac{1}{2(r-1)} \sum_{(i,j):i<j} \sum_{k,\ell} \min\{\lambda_{ik}, \lambda_{j\ell}\} \Tr Q_{ik}Q_{j\ell}
\end{cases}
\]

2. Strengthening the states-dependent factors

3. Testing composite hypotheses:

\[
\rho^{\otimes n} \text{ VS } \sum_i q_i \sigma_i^{\otimes n} \quad (\text{or, } \int \sigma^{\otimes n} \, d\mu(\sigma))
\]


Thank you !