Limits on storage of Q info in a volume of space

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Based on joint work with S. Flammia, M. Kastoryano, I. Kim; arXiv:1610.?????

What is Code?

- Scheme of storing/processing information:
  - Basic Rule = Digitize Errors:
    If σ^x and σ^z-errors on a qubit are correctable,
    then arbitrary error on that qubit is correctable.
- Foundation of feasibility of large scale quantum computing
- Useful toy models for topological order
- Fresh viewpoint on field theories with holographic dual
- The information must be redundant.
  - i.e., There are many ways to access the information.

Limited by linearity of QM

- 0000000000 vs 1111111111
  - Very redundant, but will not work under QM
  - Think of superposition: Dead Cat vs Live Cat
- The same information must be accessible in many ways
  - Polarization is accessible through any spin,
  - But, relative amplitude requires $\prod_i \sigma_i^x$, no other operator.
- But, no-cloning theorem implies
  - It is impossible to have 2 sets of operators of disjoint support that enables access to the information.

To Topological Order

- Capable of correcting local errors
~ Robust Degeneracy
~ Transformation within ground space by global operators
~ Only does matter the topology, not exact shape, of the operator support.

- Axioms of Algebraic Theory of Anyons
(Modular Tensor Category, Modular Functor, TQFT)
  - Semi-simplicity
  - Finitely many simple objects
  - Pentagon & Hexagon equations for F- and R-matrix.
  - Non-degeneracy of S-matrix

WHY?

Robust Degeneracy ~ Error Correcting Code

- $H = \sum_j h_j + \lambda \sum_j v_j$ where λ is small.

In perturbation theory,
all matrix elements
$\langle \psi_i | V | \psi_j \rangle$
should be Kronecker delta.

Matrix element to vanish is
the Knill-Laflamme condition.

Caution: QECC is the property of the state,
While the gap is the property of the Hamiltonian

Definitions

- A ***code*** is a subspace: set of allowed states
- A subset of qubits is ***correctable*** if the global state is recoverable from the erasure of those qubits.
- ***Code distance*** is the least number of qubits whose erasure cannot be corrected.

Bravyi-Poulin-Terhal, H-Preskill bounds in 2D

$H = -\sum_j \mathrm{P}_j$

where $[\mathrm{P}_j, \mathrm{P}_k] = 0$, $P_j^2 = P_j$, and $\Pi_{\mathrm{GS}} = \prod_j \mathrm{P}_j$

- $k\, d^2 \leq c\, n$
  - k = log(degeneracy)
  - d = code distance
  - n = #(qubits)
- $\tilde{d}\, d \leq c\, n$
  - $\tilde{d}$ = a region size that can support all logical operators
  - (logical operators = those act within the ground space)

To Topological Order

$$k\, d^2 \le c\, n$$
$$\tilde{d}\, d \le c\, n$$

For commuting H

- Almost an axiom:
  The degeneracy on 2-torus = #(anyon types)
- Accepting that any topological system's minimal operator for the ground space is at least "string,"

  which means $d \sim L$ and $n \sim L^2$.
- Then, k is *bounded*,
  and all the other operators must also be string-like.
- How general are these bounds?
  - Commuting Hamiltonians almost never appear in realistic models.
  - Only in terms of states?
  - All gapped systems?

Approximate Q Error Correction

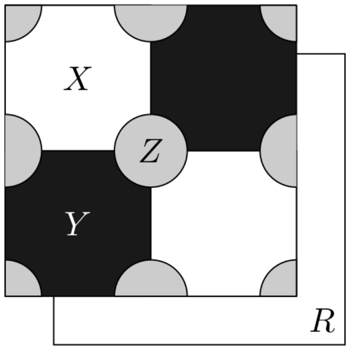
- The recovery does not have to be perfect.
- $$\mathfrak{Fidelity}(\mathcal{R} \circ \mathcal{N}(\rho), \rho) \geq 1 - \epsilon$$
- In some scenario, AQEC performs better
  - No exact code can correct $n/4$ arbitrary errors,
  - While some AQEC scheme can correct $n/2$ errors. [C. Crepeau, D. Gottesman, A. Smith (2005)]
    - This scheme uses random classical subroutine.

Our result 1

No Hamiltonian involved

- In 2D, any system with a (ground) space admitting sufficiently faithful string operators on width-ℓ strip, can only have

$$\dim \Pi_{GS} \le \exp(c\,\ell^2)$$



Sufficiently Faithful:

For every unitary logical operator U there is a string operator V such that

$$||\,(U-V)\Pi_{\mathrm{GS}}|| \le \frac{1}{5\cdot 72^4}$$

Our result 1

$$\dim \Pi_{GS} \leq \exp(c\,\ell^2)$$

- Optimal up to the constant c.
  - Bring ℓ^2copies of the toric code.
- Assumes the underlying lattice has 1 qubit per unit area.
- If not a qubit, redefine the unit length.
- If not finite-dimensional, this bound blows up.

Our result II

Error Recovery

- Assumption: Every region of size $< d$ allows recovery within ℓ-neighborhood of the region up to error δ.
- $$\left(1 - c\sqrt{\frac{n\delta}{d}}\right) k\, d^2 \leq c' n\, \ell^4$$
- There is a subset of the lattice containing $\tilde{d}$ qubits such that

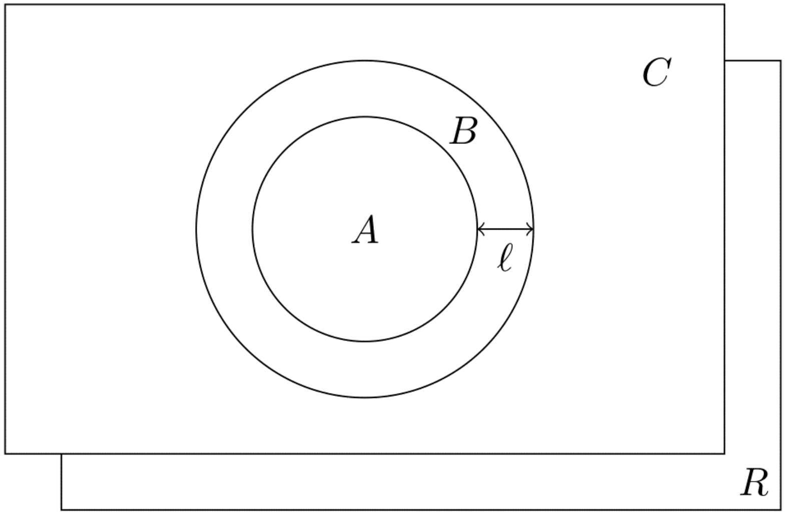
$$d\,\tilde{d} \leq c\, n\, \ell^2$$

and it can support all logical operators to accuracy $O(\sqrt{n\,\delta/d})$

- $\delta = \delta(\ell)$ decays exponentially for the ground space of a gapped Hamiltonian whose quantum phase can be represented by a commuting Hamiltonian

Why local recovery?

- Intuition from topologically ordered system



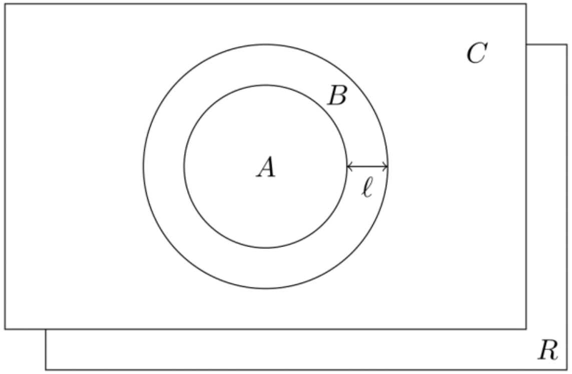
- If errors occur in A, then excitations will be in AB.
- Correction = Push the excitations towards the center.

Information Disturbance Tradeoff & Decoupling Unitary

- $\inf_{\mathcal{R}} \sup_{\rho} \mathfrak{B}(\rho^{ABCR}, \mathcal{R}_B^{AB}(\rho^{BCR}))$

$$= \inf_{\omega} \sup_{\rho} \mathfrak{B}(\omega^A \rho^{CR}, \rho^{ACR})$$

$$= \inf_{\omega,U} \sup_{\rho} \mathfrak{B}(\omega^{AB'} \rho^{B''CR},\ U_B^{B'B''} \rho^{ABCR} U_B^{B'B''})$$



$\mathfrak{B} = \sqrt{1 - \mathfrak{Fidelity}}$ is a metric.

Kretschmann, Schlingemann, Werner (2008)
Beny, Oreshkov (2010)

A region is recoverable from erasure, if and only if it is decoupled from the rest and independent of the code state

Logical operator avoidance

- Let $\mathcal{R}$ be the recovery map, and define

$$V^{BC} = \mathcal{R}^*(U^{ABC})$$

$$\begin{aligned}
\left\|\Pi(U^{ABC} - V^{BC})\Pi\right\| &= \sup_{\rho^{ABC}} \left|\mathrm{Tr}(\rho^{ABC}(U^{ABC} - V^{BC}))\right| \\
&= \sup_{\rho^{ABC}} \left|\mathrm{Tr}(\rho^{ABC}U^{ABC} - \mathcal{R}_B^{AB}(\rho^{BC})U^{ABC})\right| \\
&= \sup_{\rho^{ABC}} \left|\mathrm{Tr}((\rho^{ABC} - \mathcal{R}_B^{AB}(\rho^{BC}))U^{ABC})\right| \\
&\leq \sup_{\rho^{ABC}} \left\|\rho^{ABC} - \mathcal{R}_B^{AB}(\rho^{BC})\right\|_1 \left\|U^{ABC}\right\| \\
&\leq \epsilon \left\|U^{ABC}\right\|.
\end{aligned}$$

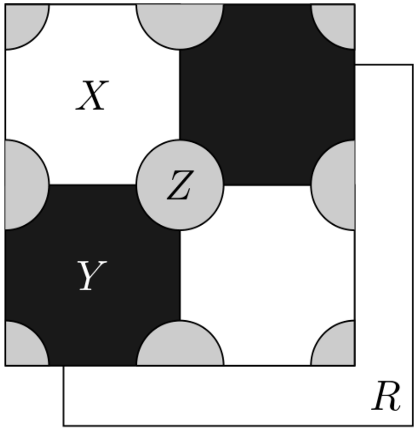
So easy! Makes us wonder why previously done some other way.
Good example where argument gets easier more generally.

Logical operator avoidance converse

- If A avoids all logical operators,
  then A is decoupled from any external system that is entangled with the code subspace.
  Hence, A is correctable.

- Pf) $U_{AB}\ \rho^{ABR}\ U_{AB}^* \simeq V_B\ \rho^{ABR}\ V_B^*$
- Take Haar average by varying U_{AB} to obtain maximally mixed code state.
- But the maximally mixed code state cannot have any correlation with external R.

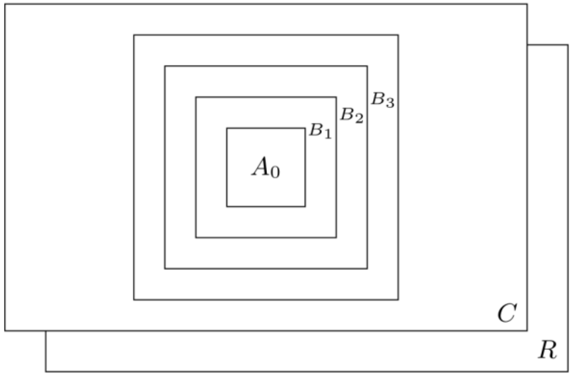
Dimension bound



- Y avoids logical operators $\Rightarrow \left|\rho^{YR} - \rho^{Y}\rho^{R}\right|_1 \leq \epsilon$.
- X avoids logical operators $\Rightarrow \left|\rho^{XR} - \rho^{X}\rho^{R}\right|_1 \leq \epsilon$.
- $I_\rho(Y\!:R) + I_\rho(X\!:R) \leq \mathrm{O}(\epsilon)\log(|R|/\epsilon)$
- Choose the maximially entangled code state with R.
- $(1 + O(\epsilon\log\epsilon\,))k \leq S\left(\rho^{Z}\right) \leq O\left(\ell^2\right)$.

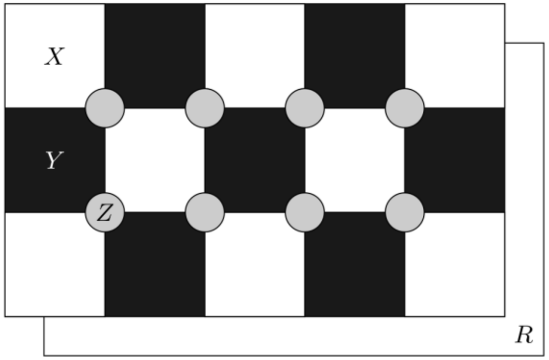
QED.

Proof of Tradeoff bounds



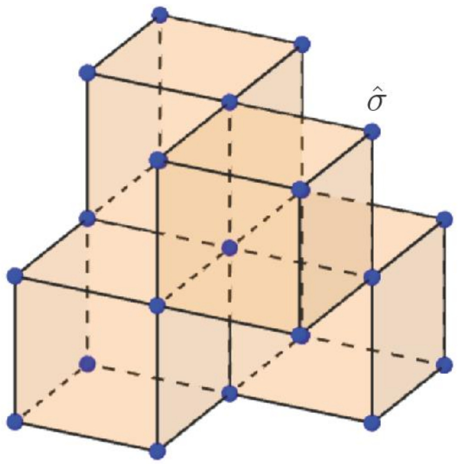
- If A is correctable and
  its boundary is correctable,
  then the union is also correctable.
- A large square is correctable
  [Bravyi,Poulin,Terhal (2010)]

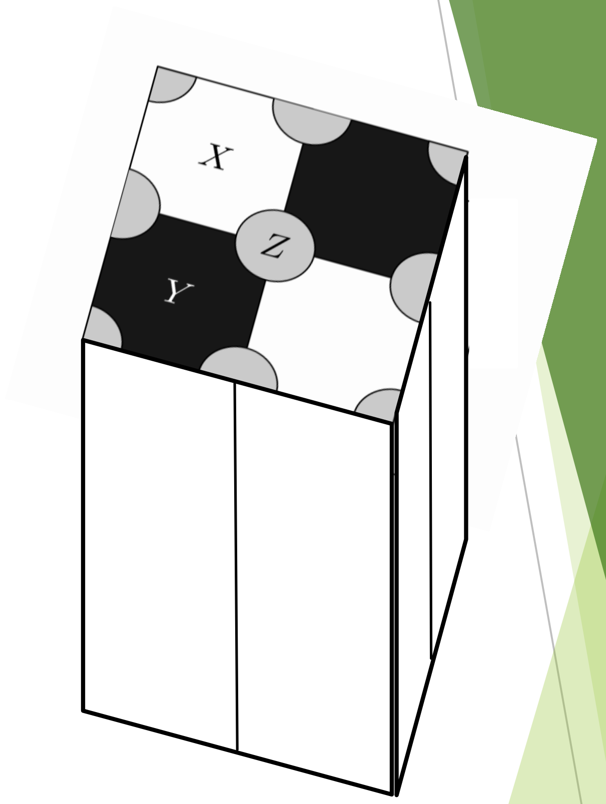
- If A is locally correctable,
  B is correctable,
  and they are separated,
  then their union is also correctable.



- Finally, apply the previous technique.
- Everything with inequality.
- Were it not for the Bures distance,
  the bound would be too weak to be meaningful.

Higher dimensions





$$\left(1 - c\sqrt{n\delta/d}\right) k d^{\frac{2}{D-1}} \leq c' n \ell^{\frac{2D}{D-1}}$$

Divide the whole lattice into checkerboard

$$k \leq O(\ell^2 L^{D-2})$$

Flexible logical operators on hyperplanes

Summary

- Introduced *locally correctable codes*
  (Every region of size less than d admits local recovery map up to accuracy δ.)
  with applications to topologically ordered systems
- Characterized Correctability via
  1. Closeness to product state upon erasure of buffer
  2. Existence of the decoupling unitary
  3. Logical operator avoidance
- Derived tradeoff bounds
  $\left(1 - c\sqrt{\frac{n\delta}{d}}\right) k\, d^2 \le c' n\, \ell^4$ and $d\, \tilde{d} \le c\, n\, \ell^2$
- Ground state degeneracy of 2D system is finite
  if string operators well approximates the action within ground space.