

# Emergent Pfaffian Relations in Quasi-Planar Models

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joint works with:  
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# 1 a. Intro: A Phenomenon of Emergent Planarity

## I. A general feature of planar Ising spin systems (at zero mag. field)

A **planar Ising model**:  $\mathcal{G} = (V(\mathcal{G}), \mathcal{E}(\mathcal{G}))$  a finite planar graph,

$$Z_{\beta, h=0} = \sum_{\sigma \in \{-1, +1\}^{\mathcal{G}}} e^{\beta \sum J_{x,y} \sigma_x \sigma_y}$$

and  $\langle \dots \rangle$  the corresponding equilibrium state average.

**Theorem** (\*) For any such system of Ising spins on a planar graph with a connected boundary segment  $\Gamma$ , and any collection of **boundary sites**  $\{x_1, \dots, x_{2n}\} \subset \Gamma$



$$\left\langle \prod_{j=1}^{2n} \sigma_{x_j} \right\rangle = \sum_{\text{pairings } \pi} \varepsilon(\pi) \prod_{j=1}^n \langle \sigma_{x_{\pi(2j-1)}} \sigma_{x_{\pi(2j)}} \rangle \equiv Pf(S_2(x_j, x_k))$$

where  $\varepsilon(\pi) = \pm 1$  is the pairing's parity, relative to the boundary's cyclic order.

\* Realized in increasing generality –

for graphs with a regular transfer matrix: Schultz-Mattis-Lieb '64,  
in the above form: Groeneveld-Boel-Kasteleyn '78.

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II. The above relation is limited to planar models. Our main goal here is to explain the emergence of such relations in the scaling limits of 2D models with **non-planar interactions**, at their critical points (an example of “universality” in critical behavior).

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## 1 b. Preliminary remarks - i. Pfaffians in Physics

Pfaffians showed earlier in statistical mechanics in the **partition functions** in certain exactly solvable models, on **planar** graphs (Kasteleyn, Fisher, Temperley '61-'63) :

- **Dimer cover:**

$$\# \text{Dimer Covers } (\mathcal{G}) = \text{Pf}(A) = \det(A)^{1/2}$$

$A$  – the Kasteleyn matrix

- **Planar Ising model:**

$$Z_{\beta, h=0} = \sum_{\sigma \in \{-1, +1\}^{\mathcal{G}}} e^{\beta \sum J_{x,y} \sigma_x \sigma_y} = \det(\mathbb{1} - KW)_{\mathcal{E}_0 \times \mathcal{E}_0}^{1/2}$$

$K, W$  – the Kac-Ward matrices

The dimensions of the matrices (or triangular arrays) appearing in the partition functions is of the order of the graph. In contrast, in the Pfaffian relations

$$S_{2n}(x_1, \dots, x_{2n}) := \left\langle \prod_{j=1}^{2n} \sigma_{x_j} \right\rangle = \text{Pf}(S_2(x_j, x_k))_{2n \times 2n}$$

the matrix dimensions correspond to the number of particles involved in the given correlation function.

The Pfaffian structure of correlations is a characteristic of **non-interacting fermions** (for which it holds in any dimension). As such, it is indicative of the model's **integrability**.

- The relation

$$\left\langle \prod_{j=1}^{2n} \sigma_{x_j} \right\rangle = \sum_{\text{pairings } \pi} \varepsilon(\pi) \prod_{j=1}^n \langle \sigma_{x_{\pi(2j-1)}} \sigma_{x_{\pi(2j)}} \rangle \equiv \text{Pf}(S_2(x_j, x_k))$$

is valid for boundary spin correlation functions on **amorphous** planar graphs, and arbitrary sets of pair couplings (not limited to ferromagnetic).

However, for rather simple reasons this relation **does not hold** for the spin correlation functions **in the bulk**, nor for non-planar models (Boel-Kast. '78)

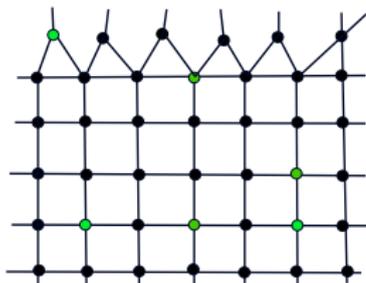
- The relation does however extend to correlation functions of the **order-disorder** operators (which will be presented below).
- Both Pfaffian relations (of boundary spin correlations, and more general of order-disorder variables) have counterparts in monomer correlation functions of planar dimer cover models. (Priezzhev- Ruelle 08, Giuliani-Jauslin-Lieb '15, A-LainzValcazar-Warzel '16)
- Our proof & explanation of the relation (ADTW'16) utilizes the **random current representation**.

## 2 a. The Ising model – the spin perspective

Ising spins on a general graph  $\mathcal{G}$ :

$$\sigma : \mathcal{G} \rightarrow \{-1, +1\}$$

$$H(\sigma) = - \sum_{(x,y) \in \mathcal{E}} J_{x,y} \sigma_x \sigma_y - h \sum_{x \in \mathcal{G}} \sigma_x$$



Gibb's equil. measure

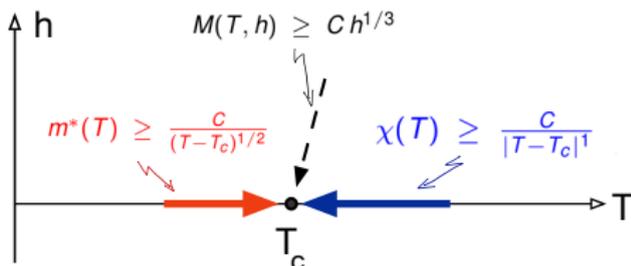
$$\Pr_{\Lambda}(\sigma) = e^{-\beta H_{\Lambda}(\sigma)} / Z_{\Lambda}$$

$$Z_{\Lambda} = \sum_{\sigma \in \{-1,1\}} e^{-\beta H_{\Lambda}(\sigma)}$$

The spontaneous magnetization:

$$m^*(T) \equiv M(T, h = 0+) := \langle \sigma_x \rangle_{T, h=0+}$$

$$\text{is } \begin{cases} 0 & T > T_c \\ > 0 & T < T_c \end{cases}$$



Phase diagram for

$$\langle \cdot \rangle = \lim_{\Lambda \nearrow \mathcal{G}} \mathbb{E}_{\Lambda}[\cdot]$$

[On transitive graphs the corresponding critical exponents are bounded by their mean field values (ABF'87):

$$\gamma \geq 1, \quad \beta \leq 1/2, \quad \delta \geq 3.]$$

## 2 b. The model's Random Current representation

The (ferr.) Ising spin system on a graph  $\mathcal{G}$  of edge set  $\mathcal{E}$  (finite subsets  $\Lambda \subset \mathcal{G}$ ) is:

$$\sigma : \mathcal{G} \mapsto \{-1, 1\}, \quad \Pr_{\Lambda}(\sigma) = \frac{e^{-\beta H_{\Lambda}(\sigma)}}{Z_{\Lambda}}$$

with  $H(\sigma) = -\sum_{(x,y) \in \mathcal{E}} J_{x,y} \sigma_x \sigma_y - h \sum_{x \in \mathcal{G}} \sigma_x$ ;  $J_{x,y} \geq 0$  (ferromag. interaction)

The Random Current representation (starting from the *high temp. exp.*, as GHS did)

$\mathbf{n} : \mathcal{E} \mapsto \{0, 1, 2, \dots\}$   $\partial \mathbf{n} := \{x \in \mathcal{G} : (-1)^{\sum_y n_{x,y}} = -1\}$  - the set of sources

weights:  $w(\mathbf{n}) := \prod_{b \in \mathcal{E}} (\beta J_b)^{n_b} / n_b!$  with "b" an alternative symbol for  $(x,y) \in \mathcal{E}$

*Basics relations (for  $h = 0$ ):*

$$Z := \sum_{\sigma} e^{-\beta H(\sigma)} = \sum_{\mathbf{n} : \partial \mathbf{n} = \emptyset} w(\mathbf{n})$$

*pictorially:*

$$Z = \sum \text{[diagrams of closed loops]}$$

and for any  $A \subset \mathcal{G}$ :

$$\langle \prod_{x \in A} \sigma_x \rangle = \sum_{\mathbf{n} : \partial \mathbf{n} = A} w(\mathbf{n}) / Z$$

$$\langle \sigma_x \sigma_y \rangle = \frac{\sum \text{[diagrams with red line between } x \text{ and } y \text{]}}{\sum \text{[diagrams]}}$$

### 3 b. Critical behavior above the upper critical dimension

$$\langle \sigma_{x_1} \dots \sigma_{x_4} \rangle = \frac{\sum_{\text{diagrams}}}{\sum_{\text{diagrams}}}$$

This yields a suggestive explanation of the phenomenon of **upper critical dimension**: in high dimensions (as it turns out  $d > 4$ ), at large separations:

$$\langle \sigma_{x_1} \dots \sigma_{x_4} \rangle \approx [\langle \sigma_{x_1} \sigma_{x_2} \rangle \langle \sigma_{x_3} \sigma_{x_4} \rangle + \langle \sigma_{x_1} \sigma_{x_3} \rangle \langle \sigma_{x_2} \sigma_{x_4} \rangle + \langle \sigma_{x_1} \sigma_{x_4} \rangle \langle \sigma_{x_2} \sigma_{x_3} \rangle] [1 + o(1)]$$

**Theorem** (A 81) For the n.n. Ising models on  $\mathbb{Z}^d$  in  $d > 4$ , if for some  $\kappa(\delta) \rightarrow \infty$  the scaled correlation functions converge (pointwise for  $x_1, \dots, x_{2n} \in \mathbb{R}^d$ )

$$S_{2n}(x_1, \dots, x_{2n}) = \lim_{\delta \rightarrow 0} \kappa(\delta)^{2n} \langle \prod_{j=1}^{2n} \sigma_{[x_j/\delta]} \rangle_{T_c}$$

then the limiting functions satisfy

$$S_{2n}(x_1, \dots, x_{2n}) = \sum_{\text{pairings } \pi} \prod_{j=1}^n S_2(x_{\pi(2j-1)}, x_{\pi(2j)})$$

Under the above conditions also (A-Barsky-Fernandez '87) :

$$\gamma = 1, \quad \beta = 1/2, \quad \delta = 3.$$

## 4. A key stochastic geometric relation

Defining  $u_4(x_1, \dots, x_4)$  so that:

$$\langle \sigma_{x_1} \sigma_{x_2} \sigma_{x_3} \sigma_{x_4} \rangle = \left[ \langle \sigma_{x_1} \sigma_{x_2} \rangle \langle \sigma_{x_3} \sigma_{x_4} \rangle + \langle \sigma_{x_1} \sigma_{x_3} \rangle \langle \sigma_{x_2} \sigma_{x_4} \rangle + \langle \sigma_{x_1} \sigma_{x_4} \rangle \langle \sigma_{x_2} \sigma_{x_3} \rangle \right] + u_4(x_1, \dots, x_4)$$

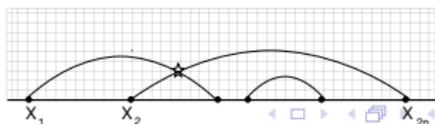
we have:

**Lemma:** For any Ising model on a finite graph:

$$u_4(x_1, \dots, x_4) = -2 \langle \sigma_{x_1} \sigma_{x_2} \rangle \langle \sigma_{x_3} \sigma_{x_4} \rangle \text{Prob} \left( C_{n_1+n_2}(x_1) \ni x_2, x_3, x_4 \mid \begin{matrix} \partial n_1 = \{x_1, x_2\} \\ \partial n_2 = \{x_3, x_4\} \end{matrix} \right)$$

**Note:**

- In situations where  $|u_4(x_1, \dots, x_4)| / \langle \sigma_{x_1} \sigma_{x_2} \sigma_{x_3} \sigma_{x_4} \rangle \rightarrow 0$  one gets **Gaussian limits**
- If for intertwined pairs:  $\text{Prob}(\dots) \rightarrow 1$ , then one gets a **fermionic expression**.
- The argument has a simple extension to all even- $n$  **boundary correlation functions** (ADTW).
- Important here are not just the statistics, but the apparent **“free-ness”** (or integrability) of the model.



## An interesting contrast

For  $d > 4$  the critical correlations  $S_{2n}(x_1, \dots, x_{2n}) = \kappa(\delta)^{2n} \langle \prod_{j=1}^{2n} \sigma_{[x_j/\delta]} \rangle_{T_c}$

satisfy 
$$S_{2n}(x_1, \dots, x_{2n}) \approx \sum_{\pi} \prod_{j=1}^n S_2(x_{\pi(2j-1)}, x_{\pi(2j)})$$
 Gauss-Wick rule (Aiz 81)

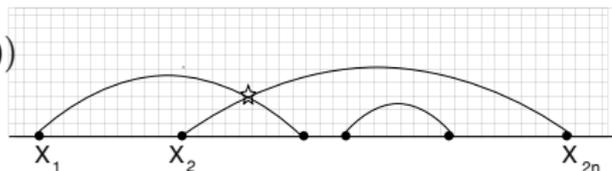
with equality in the scaling limit ( $\delta \rightarrow 0$ ,  $\kappa(\delta) \rightarrow \infty$  adjusted so the limit exists),

In  $d = 2$  dimensions for any ferromag. Ising model on a **planar** graph, with a connected boundary segment, the **boundary fields** satisfy (SML 65, McCoy-Wu'73, GBK 78):

$$S_{2n}(x_1, \dots, x_{2n}) = \sum_{\pi} \varepsilon(\pi) \prod_{j=1}^n S_2(x_{\pi(2j-1)}, x_{\pi(2j)})$$

Fermi-Wick rule  
 $x_j \in [0, \infty) \times \mathbb{R}^{d-1}$

$$= \text{Pf}(S_2(x_j, x_k))$$



Curiously, both relations have a relatively simple explanation through the “random current representation”. Using it, the **Pfaffian** structure of correlations (on which more can be read in Chelkak-Cimasoni-Kassel '15) appears as a consequence of elementary topological arguments (ADTW).

## 5. Emergent planarity

“Almost planar” – finite-range models on planar graphs.

For a class of such models we have the following statement of **emergent planarity**.

**Theorem** (ADTW '16) *In any **finite range** ferromagnetic Ising model in  $\mathcal{G} = \mathbb{Z} \times \mathbb{Z}_+$  whose couplings  $J$  are: i) translation invariant, ii) acyclic, iii) invariant under reflections: for any cyclicly ordered  $(x_1, \dots, x_{2n}) \in \partial\mathcal{G} := \mathbb{Z} \times \{0\}$*

$$\langle \sigma_{x_1} \cdots \sigma_{x_{2n}} \rangle_{\beta_c} = \text{Pf}_n \left( [\langle \sigma_{x_i} \sigma_{x_j} \rangle_{\beta_c}]_{1 \leq i, j \leq 2n} \right) (1 + o(1)) \quad (1)$$

where  $o(1)$  is a quantity tending to zero with the smallest distance in  $\mathcal{G}$  between any two  $x_j$ .

In the stochastic geometric argument the effective **planarity** emerges due to the critical random currents' **fractal nature** (at  $\beta_c$ ).

Related universality results **-stability of the law under weak perturbations-** were previously derived using rigorous (perturbative) renormalization arguments by Pinson-Spencer and Giuliani-Greenblatt-Mastropietro '12.

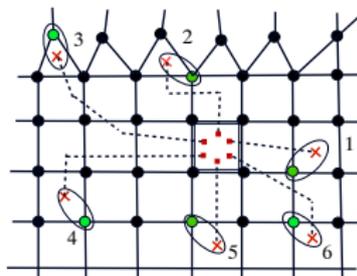
## 6. Order-disorder operators

**A natural question:** Does the fermionic structure extend to variables in the bulk?

**Our answer:** “Yes / but”: a natural extension is found in the order-disorder operators.

$\ell_j$ : dual lines linking sites of  $\{x'_j\}$   
with  $x_0^* \in \mathcal{G}^*$  (the grand central).

Coupling-reversing transform's:  
 $(R_\ell J)_{x,y} = -J_{x,y}$   
for edges  $\{x, y\}$  crossed by  $\ell$ .



The “order - disorder” variables  $\tau_{\hat{x}} = \sigma_x \mu_{x'}$  are defined by:

$$\langle \prod_{j=1}^{2n} \tau_{\hat{x}_j} \rangle := \sum_{\sigma} \left( \prod_{j=1}^{2n} \sigma_{x_j} \right) e^{-\beta R_{\ell_1} \dots R_{\ell_j} \dots H(\sigma)} / Z$$

Of particular interest:  
 $\tau_j$  for neighboring pairs  
 $\hat{x}_j = (x_j, x'_j) \in \mathcal{G} \times \mathcal{G}^*$ .

**Theorem 4** (ADTW) In planar Ising models, of pair interaction  $\mathcal{J}$  with  $Z_{\mathcal{G}}(\mathcal{J}) \neq 0$ , for any collection of “order - disorder” variables labeled cyclicly in terms of the disorder lines

$$\langle \prod_{j=1}^{2n} \tau_{\hat{x}_j} \rangle = \sum_{\text{pairings } \pi} \varepsilon(\pi) \prod_{j=1}^n \langle \tau_{\hat{x}_{\pi(2j-1)}} \tau_{\hat{x}_{\pi(2j)}} \rangle \equiv \text{Pf} \left( \langle \tau_{\hat{x}_j} \tau_{\hat{x}_k} \rangle \right). \quad (2)$$

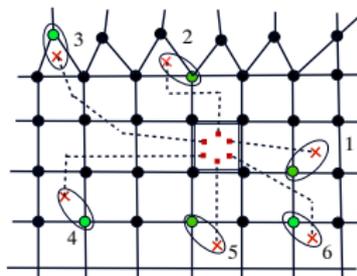
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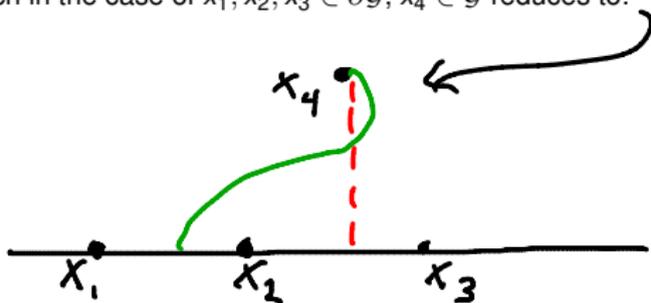
## 6 b. 'Order-disorder' operators' stochastic geometric interpretation

i) In terms of random currents:

$$\langle \tau_{\hat{x}_1} \tau_{\hat{x}_2} \rangle = \sum_{\substack{\partial n_1 = \{x_1, x_2\} \\ n_2 = \emptyset}} \frac{w(n_1)}{Z} \frac{w(n_2)}{Z} (-1)^{(n_1, \gamma_{1,2})} \mathbb{1}[\text{Diagram}]$$

↪  $n_1 = n_2 = 0$

which in the case of  $x_1, x_2, x_3 \in \partial \mathcal{G}$ ,  $x_4 \in \mathcal{G}$  reduces to:



ii) In terms of the Kac-Ward ("parafermionic") amplitudes

$$\begin{aligned} \langle \tau_{\hat{x}_1} \tau_{\hat{x}_2} \rangle &= e^{i\angle(\bar{x}_1, \bar{x}_2)} \langle \bar{e}_2 | (\mathbb{1} - KW)^{-1} | e_1 \rangle \\ &= e^{i\angle(\bar{x}_1, \bar{x}_2)} \sum_{\gamma: e_1 \rightarrow \bar{e}_2} \chi_{KW}(\gamma) e^{i \int_{\gamma} d\text{Arg}(e)/2} \end{aligned}$$

## 7. Related observations, and questions

1) The **order-disorder** operators form the Kaufman spinors, and are key elements in the Kadanoff - Ceva list.

Their product also yields the energy density operator. Through this relation, the above fermionic rule yields yet another intuitive explanation, a-la Kadanoff, of some of the (already well known) **critical exponents**, e.g.:

- the energy- energy correlations decay in 2D as  $1/r^2$   
(and hence the energy density has, in 2D, a logarithmic cusp at  $T_c$ ).
- boundary spin correlators decay as  $1/r$ , etc.

Can this structure be understood more robustly (universality, etc)?

2) **Emergent planarity**: There is still room for a more complete mathematical grasp of the stochastic geometry of the critical models. This may add some robust insight on the emergent planarity **at criticality** in two dimensional models with **non-planar** interactions / weights, supplementing the (perturbative) renormalization group analysis.

3) Among emergent structures/features of physical systems:

- fermionic excitations in classical Ising systems
- particles as collective excitations (including Majorana fermions, etc.)
- topological states of matter
- A timid question: could one day also the laws of QM be viewed as an emergent feature?

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Thank you for your attention.