Matrix product approximations to multipoint functions in two-dimensional conformal field theory

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Goal

• Understand the entanglement structure of quantum field theories using tensor network methods
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• Tensor networks model the entanglement properties of many body systems and are successfully applied in condensed matter physics
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• Tensor networks model the entanglement properties of many body systems and are successfully applied in condensed matter physics

• What about quantum field theories?

• States of the quantum field theory and tensor network states live in different Hilbert spaces: how to measure closeness?
How to approximate a quantum field theory?
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- Focus on physical quantities: correlation functions
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- If tensor networks can approximately reproduce correlation functions of quantum field theories, then we can use them to understand the entanglement structure of quantum field theories.
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• If tensor networks can approximately reproduce correlation functions of quantum field theories, then we can use them to understand the entanglement structure of quantum field theories.

• Start with simplest interesting class of quantum field theories: 1+1 dimensional unitary Conformal Field Theories (a quantum field theory defined on the circle with conformal symmetry)
Recap: Matrix product states

- Tensor network states for spin chains:
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- Tensor network states for spin chains: maximally entangled pairs of (bond) dimension $D$ are placed between the physical particles (physical dimension $d$)
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Recap: Matrix product states

- Tensor network states for spin chains: maximally entangled pairs of (bond) dimension $D$ are placed between the physical particles (physical dimension $d$) and are contracted by a $D \times D \times d \times d$ dimensional tensor.

- Correlation functions can be computed efficiently (in $D$) and reduce to the computation of a sequence of completely positive maps on matrices of dimension $D$. 

![Diagram of tensor network states for spin chains.](image-url)
Main result

Correlation functions of 1+1 dimensional unitary Conformal Field Theories can be arbitrarily well approximated by correlations functions of Matrix Product states.
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Scaling of Parameters:
number of fields $n$, UV cutoff $d$ (measured in terms of energy), approximation error $\varepsilon$, $C$ constant depending on CFT (not necessarily central charge)

<table>
<thead>
<tr>
<th>scaling of bond dimension</th>
<th>fixed $n$, UV cutoff $d$</th>
<th>fixed $\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log(D)$</td>
<td>$\sim \log(1/\varepsilon) C \frac{n}{d}$</td>
<td>$\sim \sqrt{Cn}$</td>
</tr>
</tbody>
</table>
Achievements & shortcomings

- We obtain a sequence of explicit Tensors describing Matrix product states, which better and better satisfy the conformal symmetry.
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• Uses the language of Vertex operator algebras: first introduced by Borcherds in his proof of the Moonshine conjecture.
More Symmetries: Wess-Zumino-Witten models

- In addition to conformal symmetries, WZW models possess an additional local symmetry given by an affine Lie algebra based on a simple compact Lie group.
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• These additional symmetries carry over to the MPS Tensors; leads to a group invariant MPS.
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• In addition to conformal symmetries, WZW models possess an additional local symmetry given by an affine Lie algebra based on a simple compact Lie group.

• These additional symmetries carry over to the MPS Tensors; leads to a group invariant MPS.

• Moreover, the interactions (fusion rules) are completely described already in the lowest level; the higher order Tensors are only needed to model the conformal and affine symmetries.
Proof sketch: regularization

- identify states with Hilbert-Schmidt operators on the chiral theory; field operators become linear maps: need to approximate by finite-dimensional ones

\[ \langle \phi_1(x_1), \phi_2(x_2), \ldots, \phi_n(x_n) \rangle \]
Proof sketch: regularization

- identify states with Hilbert-Schmidt operators on the chiral theory; field operators become linear maps: need to approximate by finite-dimensional ones

- a finite UV cutoff regularises the unbounded field operators and turns them into bounded operators

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< \Phi_1(x_1) \leftrightarrow \Phi_2(x_2) \ldots \Phi_n(x_n) >
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\[|x_1 - x_2| > d\]
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• techniques: use results of Wassermann for WZW models (explicit bounds), and the existence of genus-1 correlation functions for general CFTs [Zhu, Huang]
Proof sketch: renormalization

Bounded field operator
\( \phi(x) \): can change the energy by an arbitrary amount

Energy levels of the conformal Hamiltonian
Proof sketch: renormalization

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\[ \Phi_1(x_1) \]

Energy levels of the conformal Hamiltonian
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Bounded field operator \( \phi(x) \): can change the energy by an arbitrary amount

\[
\begin{align*}
\phi_1(x_1) & \quad \phi_2(x_2) \\
\vdots & \quad \vdots \\
\vdots & \quad \vdots \\
\vdots & \quad \vdots \\
\vdots & \quad \vdots \\
\vdots & \quad \vdots \\
\vdots & \quad \vdots \\
\end{align*}
\]

Energy levels of the conformal Hamiltonian...
Proof sketch: renormalization

Bounded field operator $\phi(x)$: can change the energy by an arbitrary amount.

*Precision Truncated* bounded field operator $\phi^{\text{tr}}(x)$ can only change the energy by a fixed amount.

Energy levels of the conformal Hamiltonian $\approx$
Proof sketch: renormalization

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Energy levels of the conformal Hamiltonian

$\phi_{1}(x_{1}) \quad \phi_{2}(x_{2}) \quad \approx \quad \phi_{1}^{tr}(x_{1})$
Proof sketch: renormalization

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\[
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\vdots & \quad \vdots \\
\end{align*}
\]

Energy levels of the conformal Hamiltonian

\( \phi_{1\text{tr}}(x_1) \quad \phi_{2\text{tr}}(x_2) \)

\( \approx \)

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- Our Approximations are constructive, provide rigorous error bounds and respect additional symmetries (WZW)
- Can be analysed further to understand low energy states of CFTs in quantum information theoretic terms (connection to quantum error correction?)
- Generalisation to MERA (multiscale entanglement renormalization Ansatz) seems possible and may provide better parameter scaling