Quantum chaos in many-particle systems

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Outline of the talk

• “Single”-particle quantum chaos. Single (semiclassical) limit: $\hbar \to 0$

• Many-particle quantum chaos. Double limit: $N \to \infty$, $\hbar \to 0$

Classical chaos: $\delta(t) \sim \delta(0)e^{\lambda t}$

Motivation

Quantum: $-\Delta \varphi_n = \lambda_n \varphi_n$, $\varphi_n \in L^2(M)$

BGS conjecture G. Casati, et al. 1980; O. Bohigas, et al. 1984: Correlations of $\{\lambda_n\}_{n=1}^{\infty}$ are universal, described by Random Matrix Ensembles from the same symmetry class
Semiclassical approach

Gutzwiller’s trace formula:

\[ \rho(E) = \sum_{n} \delta(E - E_n) \sim \bar{\rho}(E) + \Re \sum_{\gamma \in PO} A_\gamma \exp \left( \frac{i}{\hbar} S_\gamma(E) \right) \]

- Smooth
- Oscillating

\( A_\gamma \) stability factor,
\( S_\gamma(E) \) action of a periodic orbit \( \gamma \)

Number of periodic orbits grows exponentially with length

- No prediction on \( E_n \) from an individual \( \gamma \)
- All \( \{\gamma\} \) together \( \iff \) spectrum

- p. 4
Two-point correlation function

\[ R(\varepsilon) = \frac{1}{\bar{\rho}^2} \langle \rho(E + \varepsilon/\bar{\rho}) \rho(E) \rangle_E - 1 \]

\[ K(\tau) = \int_{-\infty}^{+\infty} R(\varepsilon) e^{-2\pi i \tau \varepsilon} d\varepsilon \approx \text{(Semiclassically)} \]

\[ \approx \frac{1}{T_H^2} \left\langle \sum_{\gamma, \gamma'} A_\gamma A^*_\gamma' e^{\frac{i}{\hbar} (S_\gamma - S_{\gamma'})} \delta \left( \tau - \frac{(T_\gamma + T_{\gamma'})}{2T_H} \right) \right\rangle_E , \]

\( T_\gamma, T_{\gamma'} \) are periods of \( \gamma, \gamma' \), \( T_H = 2\pi \hbar \bar{\rho} \) (Heisenberg time)

**Spectral correlations \iff Correlations between actions of periodic orbits**
Classical origins of universality

\[ K(\tau) = c_1 \tau + c_2 \tau^2 \ldots \]

\(c_1\) – diagonal approximation \(\gamma = \gamma'\) M. Berry 1985

Diagonal approximation

Sieber–Richter pairs

\(c_2\) – non-trivial correlations (Sieber-Richter pairs)
M. Sieber K. Richter 2001

\[ S_\gamma - S_\gamma' \sim \hbar \implies \text{Duration of encounter } \sim \tau_E = \lambda^{-1} |\log \hbar| \]

Duration of encounter

Ehrenfest time

All orders in \(\tau = \text{RMT result}\) S. Müller, et. al., 2004

– p. 6
Symbolic Dynamics

Continues flow $\implies$ Map $T$ (Poincare section)

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Phase space partition:

$$V = V_0 \cup V_1 \cup \ldots \cup V_{l-1}$$

Point in the phase space:

$$x = \ldots x_{-1}x_0 \cdot x_1x_2 \ldots; \quad x_i \in \{0, 1, \ldots, l-1\}$$

$$T \cdot x = \ldots x_{-1}x_0x_1 \cdot x_2x_3 \ldots$$

Periodic orbits $\iff [x_1x_2 \ldots x_n]$
[\gamma_1] = [AECFBEDF], \quad [\gamma_2] = [AEDFBECF]

\begin{align*}
E &= e_1e_2 \ldots e_p, \\
F &= f_1f_2 \ldots f_p
\end{align*}

Each p-subsequence of symbols from \( \gamma_1 \) appears in \( \gamma_2 \)

Locally similar but not identical \( \implies \)

Two orbits pass approximately the same points of the phase space:

\[ \| \gamma_1 - \gamma_2 \| \sim \Lambda^{-p} \]
Many-particle systems

\[ \mathcal{H} = \sum_{n=1}^{N} \frac{p_n^2}{2m} + V(x_n) + V_{\text{int}}(x_n - x_{n+1}) \]

Chaos, Local interactions, Invariance under \( n \rightarrow n + 1 \)

Two views on dynamics:

Many-particle Periodic Orbit \( d \)-dimensions

Single-particle Periodic Orbit \( Nd \)-dimensions

Q: Is the single-particle theory of Quantum Chaos applicable?
Semiclassical “Field Theory”

Continuous limit: \( n \to \eta \in \[0, \ell\], \quad x_{n,t} \to \phi(\eta, t) \)

\[ \mathcal{L} = \sum_{n=1}^{N} \frac{\dot{q}_{n,t}^2}{2m} + \kappa (x_{n,t} - x_{n+1,t})^2 - V(x_{n,t}) \implies \]

\[ \mathcal{L} = \int_{0}^{\ell} d\eta \left( \partial_t \phi(\eta, t) \right)^2 + \left( \partial_\eta \phi(\eta, t) \right)^2 - V(\phi(\eta, t)) \]

1) **PO** - are 2D toric surfaces in \( d \)-dim space (Rather than 1D lines in \( N \cdot d \)-dim)

2) **Encounters** are “rings” (Rather than 1D stretches) of “width” \( \sim \lambda^{-1} |\log \hbar_{eff}| \)
1) **Small alphabet** (does not grow with $N$)

2) **Uniqueness:** Each PO $\Gamma$ is uniquely encoded by $\mathbb{M}_\Gamma$

3) **Locality:** $r \times r$ square of symbols around $(n, t)$ defines position of the $n$’th particle at the time $t$ up to error $\sim \Lambda^{-r}$

**Encounter - repeating region of symbols**
Different types of Partner Orbits

A. Single particle partners:

Dominant iff \( T \gtrsim W_\hbar \gtrsim N \) - Single particle theory

\[
W_\hbar \sim \Lambda^{-1} \left| \log \hbar_{\text{eff}} \right| \approx \text{Width of encounter}
\]

B. Dual partners:

Dominant iff \( T \lesssim W_\hbar \lesssim N \) - Thermodynamic, short time regime
Different types of Partner Orbits

C. If $T \gtrsim W_\hbar$, $N \gtrsim W_\hbar$ i.e. $T$ and $N$ are larger than the “Ehrenfest scale”:

Note: One encounter is enough, even if time reversal symmetry is broken

B, C - Genuine many-particle Quantum Chaos!
A Lone Cat Map: $\mathbb{T}^2 \rightarrow \mathbb{T}^2$

Phase space: $q_t, p_t \in [0, 1)$, windings $m_t = (m^q_t, m^p_t) \in \mathbb{Z}$

Configuration Space

\[
\begin{pmatrix}
q_{t+1} \\
p_{t+1}
\end{pmatrix} = \begin{pmatrix}
a & 1 \\
ab & b
\end{pmatrix} \begin{pmatrix}
q_t \\
p_t
\end{pmatrix} - \begin{pmatrix}
m^q_t \\
m^p_t
\end{pmatrix},
\]

$a, b \in \mathbb{Z}$. Chaos if $|a + b| > 2$

Newton form: $\Delta q_t \equiv q_{t+1} - 2q_t + q_{t-1} = (a + b - 2)q_t - m_t$
Coupled-Cat Maps: $T^{2N} \rightarrow T^{2N}$

$S(q_t, q_{t+1}) = S_0(q_t, q_{t+1}) + S_{\text{int}}(q_t), \, q_t = (q_{1,t}, q_{2,t} \ldots q_{N,t})$

$N$ Interacting cat maps, $q_{n,t}, p_{n,t} \in [0, 1)$:

$S_0 = \sum_{n=1}^{N} S_{\text{cat}}(q_{n,t}, q_{n,t+1}) + V(q_{n,t}); \quad S_{\text{int}} = -\sum_{n=1}^{N} q_{n,t}q_{1+n,t}$

Equations of motion:

$p_{n,t} = -\frac{\partial S}{\partial q_{n,t}} \quad p_{n,t+1} = \frac{\partial S}{\partial q_{n,t+1}}$
Classical Particle-time Duality

Newtonian form:

\[ \Delta q_{n,t} = (a + b - 4)q_{n,t} + V'(q_{n,t}) - m_{n,t} \]

Discrete Laplacian:

\[ \Delta f_{n,t} \equiv f_{n+1,t} + f_{n-1,t} + f_{n,t+1} + f_{n,t-1} - 4f_{n+1,t} \]

Particle-time symmetry: \( t \leftrightarrow n \)

\( N \)-particle POs \( \{ \Gamma \} \) of period \( T \) \( \iff \) \( T \)-particle POs \( \{ \Gamma' \} \)

of period \( N \)

\[ S(\Gamma) = S(\Gamma'), \quad A\Gamma = A\Gamma' \]

\[ \{ m_{n,t} \} \] - provide symbolic encoding of POs
2D Symbolic Dynamics

\[ M_{\Gamma} = \begin{pmatrix}
  m_{1,1} & m_{2,1} & \ldots & m_{N,1} \\
  m_{1,2} & m_{2,2} & \ldots & m_{N,2} \\
  \vdots & \vdots & \ddots & \vdots \\
  m_{1,T} & m_{2,T} & \ldots & m_{N,T}
\end{pmatrix} \]

- **Small alphabet** (does not grow with \( N \))
- **Uniqueness** + \( \Gamma \) can be easily restored from \( M_{\Gamma} \)
- **Locality** \((r \times r \) square of symbols around \((n, t)\) defines approx. position of the \( n'th\) particle at the time \( t)\)

Example of Partner Orbits

$T = 50, \, N = 70, \, a = 3, \, b = 2$
All the points of $\Gamma = \{(q_{n,t}, p_{n,t})\}$ and $\bar{\Gamma} = \{ (\bar{q}_{n,t}, \bar{p}_{n,t}) \}$ are paired
Distances between paired points

\[ d_{n,t} = \sqrt{(q_{n,t} - \bar{q}_{n',t'})^2 + (p_{n,t} - \bar{p}_{n',t'})^2}, \]

Largest distances \( \sim 2 \cdot 10^{-3} \) are between points in encounters
Quantisation

Hannay, Berry (1980); Keating (1991)

\( U_N \) is \( L^N \times L^N \) unitary matrix, \( L = \hbar^{-1}_{\text{eff}} \)

Translational symmetries: \( \implies N \) subspectra
approximately of the same size = \( L^N / N \)

Gutzwiller trace formula

Rivas, Saraceno, A. de Almeida (2000)

\[
\text{Tr} \left( U_N \right)^T = \left| \det (B_N^T - 1) \right|^{-\frac{1}{2}} \sum_{\Gamma \in \text{PO}} \exp(-i2\pi LS_\Gamma).
\]

All entries are symmetric under exchange \( N \leftrightarrow T \)
Quantum Duality

\[
\text{Tr} \left( U_N \right)^T = \text{Tr} \left( U_T \right)^N
\]

Form Factor: 
\[
K_N(T) = \frac{1}{2L^N} \left\langle \left| \text{Tr} \left( U_N \right)^T \right|^2 \right\rangle
\]

For short times \( T < n_E = \lambda^{-1} \log L \), \( N \sim L^T \)

Regime dual to universal:
\[
K_N(T) = L^{T-N} K_\beta(TN/L^T)
\]

In particular for very short times \( L^T/T < N \), \( K_\beta \approx 1 \)
\[
K_N(T) \approx L^T/L^N
\]

Short time exponential growth instead of linear \( TN/L^N \)
Summary

\[ t_E \sim \log h \quad \text{No partner periodic orbits} \]

\[ t_H \quad \text{Universal regime} \]

\[ \mathcal{K} = \frac{1}{TN} \left\langle \left| \text{Tr} \left( U_N \right)^T \right| ^2 \right\rangle \]

**Duality:**

\[ \mathcal{K}(N, T) = \mathcal{K}(T, N) \]
Many-particle Semiclassical Programm

Single-particle structure diagrams:

\[
\begin{align*}
\{ e_1, e_2 \} = \overset{\text{N×T}}{=} \{ E_1^{(1)}, E_2^{(2)}, E_1^{(3)} \}
\end{align*}
\]

Distinguished by order of encounters

Many-particle structure diagrams:

Distinguished by order and winding numbers \( \omega \) of encounters!