Perturbation theory of non-demolition measurements

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Based on Two Articles


Phenomena: After a long sequence of measurement results $\xi_1, \xi_2, \ldots$ the state of light in the cavity is close to a photon number state.

Electron gun
Mathematical Description

Jump operators $V_\xi$, $\xi \in \sigma$, with a normalization $\sum_{\xi \in \sigma} V_\xi^* V_\xi = 1$. Probability to measure $\xi$ is $\text{Tr}(V_\xi^* V_\xi \rho)$ and the state changes as

$$\rho \rightarrow \frac{V_\xi \rho V_\xi^*}{\text{Tr}(V_\xi \rho V_\xi^*)}.$$ 

For an infinity history $\xi = \xi_1, \xi_2, \ldots$ we put $V^{(n)}(\xi) = V_{\xi_1} \ldots V_{\xi_n}$, the probability of a finite history $\xi_1, \ldots \xi_n$ is then

$$\mathbb{P}_\rho(\xi_1, \ldots, \xi_n) = \text{Tr}((V^{(n)}(\xi))^* V^{(n)}(\xi) \rho)$$

and the state evolves to

$$\rho_n(\xi) = \frac{V^{(n)}(\xi) \rho (V^{(n)}(\xi))^*}{\text{Tr}(V^{(n)}(\xi) \rho (V^{(n)}(\xi))^*)}.$$
Remarks

1. Products of i.i.d. random matrices studied by [Furstenberg, Kesten Annals of Stat. 1960] and many others with applications to 1D random Schrödinger e.g [Bougerol, Lacroix (1985)] Measurement in Quantum Mechanics also leads to study of product of matrices but with non i.i.d. measure.

2. The setting is an example of a finitely correlated state [Fannes, Nachtergaele, and Werner CMP 144].
Example $\varepsilon = 0$

For a two level system and $\sigma = \{e, g\}$ we put

$$V_e = e^{-i\varepsilon \sigma_1} \begin{pmatrix} \sqrt{p} & 0 \\ 0 & \sqrt{q} \end{pmatrix}, \quad V_g = e^{-i\varepsilon \sigma_1} \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-q} \end{pmatrix}$$

For $\varepsilon = 0$ the population of $\mathcal{N} = \sigma_z$ approaches an eigenstate,
Example $\varepsilon \neq 0$

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For $\varepsilon \neq 0$ the population of $\mathcal{N} = \sigma_z$ jumps between eigenstates,
Non-demolition case and its perturbations

For an observable $\mathcal{N}$ and a Hamiltonian $H$ with $[H, \mathcal{N}] \neq 0$ we put

$$V_{\xi}^{(\varepsilon)} = e^{-i\varepsilon H}V_{\xi}(\mathcal{N})$$

for some complex functions $V_{\xi}(\cdot)$.

In the case $\varepsilon = 0$ we have $[V_{\xi}, V_{\xi'}] = 0$ and

$$\mathbb{P}_\rho(\xi_1, \ldots, \xi_n) = \int_{\sigma(\mathcal{N})} |V_{\xi_1}(\nu)|^2 \ldots |V_{\xi_n}(\nu)|^2 d\lambda_\rho(\nu),$$

where $\lambda_\rho$ is a spectral measure of $\mathcal{N}$. Interpreting $\nu$ as an unknown we define maximum likelihood estimate

$$\hat{\mathcal{N}}_k(\xi) = \arg\max_{\nu \in \sigma(\mathcal{N})} l_k(\nu|\xi), \quad l_k(\nu|\xi) := \frac{1}{k} \sum_{j=1}^k \log |V_{\xi_j}(\nu)|^2.$$
Non-Demolition Case, $\varepsilon = 0$

For set $N \in \sigma(\mathcal{N})$ let let $\Pi(N)$ be the associated spectral projection and $S(\nu|N) = \inf_{\nu' \in N} \sum_{\xi \in \sigma} |V_{\xi}(\nu)|^2[l(\nu|\xi) - l(\nu'|\xi)]$.

**Theorem (Law of Large Numbers)**

*Suppose $\nu \to V_{\xi}(\nu)$ is injective and $V_{\xi}(\cdot)$ is continuos for all $\xi \in \sigma$, then the maximum likelihood estimator $\hat{N}_k$ converges almost surely to a random variable $\hat{N}_\infty$. For any Borel set $N \subset \sigma(\mathcal{N})$,*

$$\mathbb{P}_\rho(\xi : \lim_{k \to \infty} \hat{N}_k \in N) = \operatorname{Tr}(\Pi(N)\rho).$$

*Moreover if $N$ is a closed subset of $\sigma(\mathcal{N})$ contained in the support of the measure $\lambda_\rho$ then we have*

$$-\lim_{k \to \infty} \frac{1}{k} \log \operatorname{Tr}(\Pi(N)\rho_k) = S(\hat{N}_\infty|N), \quad \mathbb{P}_\rho - \text{almost surely.}$$
Non-Demolition Case - references

When spectrum of $\mathcal{N}$ is discrete the Law of Large Numbers was proved in

- Maassen, Kümmerer 2006
- Bauer, Bernard PRA 2011
- Mabuchi et. al. IEEE 2004

The large deviation theory

- Bauer, Benoist, Bernard AHP 2013
We make measurement times \( t_1, t_2, \ldots \) of \( \xi_1, \xi_2, \ldots \) random and distributed by Poisson distribution \( N_t \), then the evolution is

\[
\tau^s(\xi)\rho = e^{-i\varepsilon H(s-t S_t)} V_{\xi S_t} \ldots e^{-i\varepsilon H t_1} \rho e^{i\varepsilon H t_1} V_{\xi_1} \ldots e^{i\varepsilon H(s-t S_t)}.
\]

This is an unravelling of Lindblad evolution,

\[
\mathbb{E} \left[ \tau^s(\xi) \right] = \exp(s \mathcal{L}_{\varepsilon}), \quad \mathcal{L}_{\varepsilon} \rho = -i\varepsilon [H, \rho] + \sum_{\xi \in \sigma} V_{\xi} \rho V_{\xi}^* - \rho.
\]

For a sampling time \( T > 0 \) we define

\[
\hat{N}_s := \arg\max_{\nu \in \sigma(N)} \frac{1}{N_{s+T} - N_s} \sum_{\nu=\xi_1}^{N_{s+T}} I(\nu|\xi_j).
\]

To avoid dealing with overlapping data we set \( M_{jT} = \hat{N}_{jT} \), for \( j \in \mathbb{N} \) and extend the definition of \( M_t \) to all \( t \geq 0 \) by declaring it to be piecewise constant on the intervals \( [jT, (j+1)T) \).
Demolition Case

Let \( \mathcal{N} = \sum_{\nu} \nu P_\nu \) and define

\[
\mathcal{P}_\rho = \sum_{\nu \in \sigma(\mathcal{N})} P_\nu \rho P_\nu, \quad P_\nu = |\nu\rangle\langle\nu|.
\]

We define an operator \( Q \) on the range of \( \mathcal{P} \) by

\[
\varepsilon^2 Q = -\mathcal{P} \mathcal{L}_\varepsilon \mathcal{P} \mathcal{L}_0^{-1} \mathcal{P} \mathcal{L}_\varepsilon \mathcal{P}.
\]

The matrix \( Q \) defines a Markov Kernel on \( \sigma(\mathcal{N}) \) with elements

\[
\text{Tr}(P_{\nu'} Q P_\nu) = \begin{cases}
\sum_{\beta \neq \nu} \frac{|\langle \beta | H | \nu \rangle|^2}{\sum_{\xi} V_\xi(\beta) V_\xi(\nu) - 1} + \text{c.c.} & \text{for } \nu = \nu' \\
-\frac{|\langle \nu' | H | \nu \rangle|^2}{\sum_{\xi} V_\xi(\nu') V_\xi(\nu) - 1} + \text{c.c.} & \text{for } \nu \neq \nu'.
\end{cases}
\]

Let \( Y_s \) be the continuous time Markov chain generated by \( Q \) started from an initial probability distribution \( \pi_\rho(\nu) = \text{Tr}(P_{\nu\rho}) \).
Demolition Case

**Theorem (Distribution of Jumps)**

Suppose \( \nu \to V_\xi(\nu) \) is injective, and pick a positive \( I \) strictly smaller than \( \min_{\nu, \nu'} S(\nu, \nu') \). Let \( T = -\beta \log \varepsilon \), for some \( \beta > \max\{2(1 - e^{-I})^{-1}, (1 - e^{-\frac{I}{2}})^{-1}\} \). Then under \( P_{\rho}^{(\varepsilon)} \), \( \mathcal{M}_{\varepsilon^{-2}s} \) converges in law to \( Y_s \), and the posterior density matrix \( \rho_{\varepsilon^{-2}s} \) converges in law to \( P_{Y_s} \).
Thank you for your attention!