Dirac Operators with Magnetic Links

Jan Philip Solovej
Department of Mathematical Sciences
University of Copenhagen

QMath13, Atlanta October 2016

Joint work with Fabian Portmann and Jeremy Sok

Outline of Talk

1. Why Consider Zero Modes for Dirac Operator?
2. Magnetic Knots
3. Magnetic Links
4. Dirac Operators with Magnetic Links
5. The Torus of Fluxes and the Spectral Flow
6. The Effective One-dimensional Operator
7. Calculation of the Spectral Flow around the critical sets
8. Calculation of Spectral Flow for Unknots
9. Summary

Why Consider Zero Modes for Dirac Operators?

Stability of matter with magnetic fields (Fefferman 1995, Lieb-Loss-Solovej 1995) is limited due to existence of finite energy magnetic fields (Loss-Yau 1986)

$$\int \mathbf{B}^2 < \infty, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

for which the **Pauli Operator**

$$(-i\nabla - \mathbf{A})^2 - \mathbf{B} \cdot \sigma = [(-i\nabla - \mathbf{A}) \cdot \sigma]^2$$

has zero-modes, i.e., a non-trivial kernel.
We are therefore interested in understanding the kernel of the 3-dimensional Dirac operator

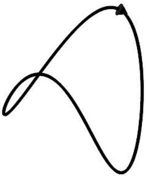
$$(-i\nabla - \mathbf{A}) \cdot \sigma$$

Magnetic Knots

A **magnetic knot** is a magnetic field with only **one field line**, similar to the Aharonov-Bohm field, except we assume it to be a closed field line:

$$\mathbf{B} = \Phi \mathbf{T} \delta_C,$$

where C is a closed oriented curve with $\mathbf{T}$ being the **unit tangent vector** and $\Phi > 0$ is the flux.



The field comes from a **singular vector potential**

$$\mathbf{B} = \Phi \nabla \times \mathbf{A}$$

with $\mathbf{A} = \mathbf{N}\delta_S$, where S is a surface with normal $\mathbf{N}$ such that $\partial S = C$.

By Seifert's Theorem such a surface always exists.

Magnetic Links

A **magnetic link** is a singular magnetic field with only finitely many (possibly interlinking) field lines, i.e., a sum of finitely many magnetic knots.

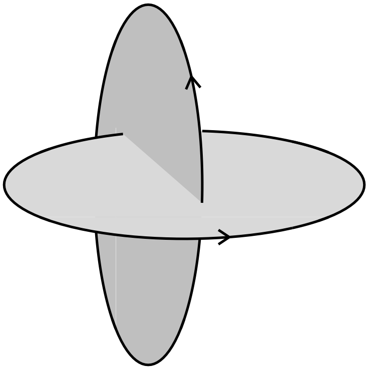


Figure: Intersecting Seifert surfaces. Hopf-link (Erdős-Solovej 2003)

Dirac Operators with Magnetic Links

To define self-adjoint **Dirac Operator** with magnetic link in the **singular gauge** introduce **domain**:

- **Phase jump** on the surface
- Appropriate **boundary condition** on the knots

Can be seen as **strong resolvent convergence** from smooth case. Write $\underline{\Phi} = 2\pi\underline{\alpha}$.

Jump condition implies that the corresponding Dirac operator $\mathcal{D}_{\underline{\alpha}}$ is periodic with period 1 in α_j.

Thus one should be able to study the **spectral flow**.

To have discrete spectrum consider $\mathbb{S}^3$. Really not important as both **spectral flow** and **kernel** of Dirac operators are **conformal invariants**.

The Torus of Fluxes and the Spectral Flow

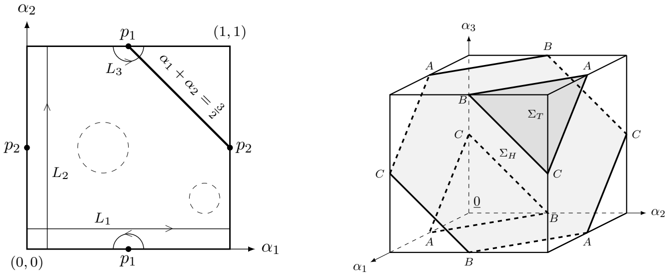


Figure: The cut torus for the Hopf 2 (left) and 3-link (right).

The Dirac Operator $\mathcal{D}_{\underline{\alpha}}$ is **norm resolvent convergent** except for $\alpha_j \to 1-$, where we will lose eigenvalues. The Spectral flow is a homotopy invariant on the tori with the critical points p_1 and p_2 or critical lines ABA, CBC, and CAC removed.

The Effective One-dimensional Operator

For a general link of knots γ_k **the critical sets** are determined by eigenvalues **disappearing at zero**. The corresponding eigenspinors **disappear on to the knot** γ_j if $\alpha_j \to 1-$ and the disappearing eigenvalues λ_n are eigenvalues of an **effective 1-dimensional operator** on the knot:

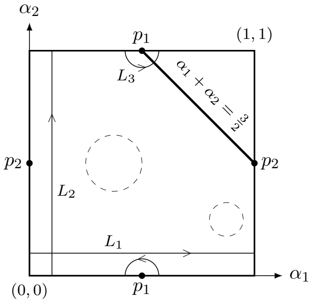
$$\lambda_n = \frac{1}{|\gamma_j|}\Big(2n\pi + \pi - \pi \mathrm{Wr}(\gamma_j) + 2\pi \sum_{k\neq j} \alpha_k \mathrm{link}(\gamma_j, \gamma_k)\Big), \quad n \in \mathbb{Z}$$

where Wr denotes the **Writhe** of the knot. Thus the critical set on the phase $\alpha_j = 1$ is given by

$$-\frac{1}{2}\mathrm{Wr}(\gamma_j) + \sum_{k\neq j} \alpha_k \mathrm{link}(\gamma_j, \gamma_k) \in \frac{1}{2} + \mathbb{Z}$$

Calculation of Spectral Flow around the critical sets

The expression for λ_n gives the spectral flow around the critical sets. **Example:**



We have $\mathsf{Sf}(L_3) = \mathsf{sgn}(\mathsf{link}(\gamma_2, \gamma_1))$. The spectral flows along L_1 or L_2 are more difficiult. We can only calculate them for unknots.

Calculation of Spectral Flow for Unknots

Spectral flow for unknot is $\lfloor \frac{1}{2} - \frac{1}{2}\mathrm{Wr}(\gamma) \rfloor$. **Proof:**

- Explicit calculation: spectral flow of great circles vanishes
- Under deformation $\gamma \to \gamma'$ spectral flow for any knot changes:

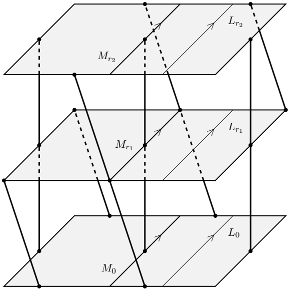
$$\lfloor \frac{1}{2} - \frac{1}{2}\mathrm{Wr}(\gamma') \rfloor - \lfloor \frac{1}{2} - \frac{1}{2}\mathrm{Wr}(\gamma) \rfloor$$

Put small circle γ_2 around γ

$$\begin{aligned} \mathrm{Sf}[L_0] &= \mathrm{Sf}[L_{r_2}] \\ \mathrm{Sf}[L_{r_2}] &= \mathrm{Sf}[M_{r_2}] \\ \mathrm{Sf}[L_0] &= \mathrm{Sf}[M_0] + 1 \end{aligned}$$

Conclusion

$$\mathrm{Sf}[M_{r_2}] = \mathrm{Sf}[M_0] + 1.$$



Summary

- We have **explicit formula** for spectral flow along **any unknot**:

$$\lfloor \frac{1}{2} - \frac{1}{2}\mathrm{Wr}(\gamma) \rfloor$$

- We have **explicit formula** for spectral flow for any closed loop on the cut-torus of fluxes for a **link of unknots**. Depends on the **writhes** of the knots and their **linking numbers**.
- For general knots, e.g., the **trefoil** we do not have a formula.
- The spectral flow is unchanged by appropriately smoothing magnetic links

An unknot with a **high spectral flow** (large writhe):