Derivation of the Maxwell-Schrödinger Equations from the Pauli-Fierz Hamiltonian

Peter Pickl

Mathematisches Institut
LMU
joint work with Nikolai Leopold

8. Oktober 2016

Overview

1. Motivation
2. Mean field limits - general remarks
3. Deriving the Hartree-Maxwell equations

Overview

1. Motivation
2. Mean field limits - general remarks
3. Deriving the Hartree-Maxwell equations

Overview

Overview

1. Motivation
2. Mean field limits - general remarks
3. Deriving the Hartree-Maxwell equations

Motivation

- Semiclassical Schrödinger equation

$$id_t\Psi_t = ((i\nabla + A_t)^2 + V_t)\Psi_t$$

  with classical A and V.
- Goal: Derive this equation from QED.
- Standard textbook argument: Heisenberg equations
  Problem: Result on expectation values only,
  to general.

Motivation

- Semiclassical Schrödinger equation

$$i d_t \Psi_t = ((i\nabla + A_t)^2 + V_t)\Psi_t$$

with classical A and V.

- Goal: Derive this equation from QED.
- Standard textbook argument: Heisenberg equations
  Problem: Result on expectation values only,
  to general.

Motivation

- Semiclassical Schrödinger equation

$$id_t\Psi_t = ((i\nabla + A_t)^2 + V_t)\Psi_t$$

  with classical A and V.
- Goal: Derive this equation from QED.
- Standard textbook argument: Heisenberg equations
  Problem: Result on expectation values only,
  to general.

Motivation

- Semiclassical Schrödinger equation

$$id_t\Psi_t = ((i\nabla + A_t)^2 + V_t)\Psi_t$$

  with classical A and V.
- Goal: Derive this equation from QED.
- Standard textbook argument: Heisenberg equations
  Problem: Result on expectation values only,
  to general.

Motivation

- Semiclassical Schrödinger equation

$$i d_t \Psi_t = ((i\nabla + A_t)^2 + V_t)\Psi_t$$

  with classical A and V.
- Goal: Derive this equation from QED.
- Standard textbook argument: Heisenberg equations
  Problem: Result on expectation values only,
  to general.

Motivation

- Semiclassical Schrödinger equation

$$id_t\Psi_t = ((i\nabla + A_t)^2 + V_t)\Psi_t$$

  with classical A and V.
- Goal: Derive this equation from QED.
- Standard textbook argument: Heisenberg equations
  Problem: Result on expectation values only,
  to general.

Skeleton of a proof

- Known fact: Photons in a coherent state interact with charges classically.
- We want a system, where photons are created by the charges.
- Problem: prove classical behaviour of back-reaction on the charges.
- Main Idea: Special system of bosons in a condensate.
  semiclassical equation via mean field limit.

Skeleton of a proof

- Known fact: Photons in a coherent state interact with charges classically.
- We want a system, where photons are created by the charges.
- Problem: prove classical behaviour of back-reaction on the charges.
- Main Idea: Special system of bosons in a condensate. semiclassical equation via mean field limit.

Skeleton of a proof

- Known fact: Photons in a coherent state interact with charges classically.
- We want a system, where photons are created by the charges.
- Problem: prove classical behaviour of back-reaction on the charges.
- Main Idea: Special system of bosons in a condensate.
  semiclassical equation via mean field limit.

Skeleton of a proof

- Known fact: Photons in a coherent state interact with charges classically.
- We want a system, where photons are created by the charges.
- Problem: prove classical behaviour of back-reaction on the charges.
- Main Idea: Special system of bosons in a condensate.
  semiclassical equation via mean field limit.

Skeleton of a proof

- Known fact: Photons in a coherent state interact with charges classically.
- We want a system, where photons are created by the charges.
- Problem: prove classical behaviour of back-reaction on the charges.
- Main Idea: Special system of bosons in a condensate.
  semiclassical equation via mean field limit.

Skeleton of a proof

- Known fact: Photons in a coherent state interact with charges classically.
- We want a system, where photons are created by the charges.
- Problem: prove classical behaviour of back-reaction on the charges.
- Main Idea: Special system of bosons in a condensate.
  semiclassical equation via mean field limit.

Mean field for the bosons: The Hartree equation

$$H = \sum_{j=1}^{N} -\Delta_j + \sum_{j=1}^{N} A_t(x_j) + N^{-1} \sum_{k<j} V(x_j - x_k)$$

Interaction „felt" by each particle of order one

$\Psi_0 = \prod_{j=1}^{N} \phi_0(x_j)$ $\quad$ $id_t \Psi_t = H_t \Psi_t$

Interaction destroys product structure.

Question:

- In which regimes: $\Psi_t \approx \prod_{j=1}^{N} \phi_t(x_j)$
- What is ϕ_t?

Mean field for the bosons: The Hartree equation

$$H = \sum_{j=1}^{N} -\Delta_j + \sum_{j=1}^{N} A_t(x_j) + N^{-1} \sum_{k<j} V(x_j - x_k)$$

Interaction „felt“ by each particle of order one

$\Psi_0 = \prod_{j=1}^{N} \phi_0(x_j)$ $\quad$ $id_t\Psi_t = H_t\Psi_t$

Interaction destroys product structure.

Question:

- In which regimes: $\Psi_t \approx \prod_{j=1}^{N} \phi_t(x_j)$
- What is ϕ_t?

Mean field for the bosons: The Hartree equation

$$H = \sum_{j=1}^{N} -\Delta_j + \sum_{j=1}^{N} A_t(x_j) + N^{-1} \sum_{k<j} V(x_j - x_k)$$

Interaction „felt“ by each particle of order one

$\Psi_0 = \prod_{j=1}^{N} \phi_0(x_j)$ $\qquad$ $id_t\Psi_t = H_t\Psi_t$

Interaction destroys product structure.

Question:

- In which regimes: $\Psi_t \approx \prod_{j=1}^{N} \phi_t(x_j)$
- What is ϕ_t?

Mean field for the bosons: The Hartree equation

$$H = \sum_{j=1}^{N} -\Delta_j + \sum_{j=1}^{N} A_t(x_j) + N^{-1} \sum_{k<j} V(x_j - x_k)$$

Interaction „felt“ by each particle of order one

$\Psi_0 = \prod_{j=1}^{N} \phi_0(x_j)$ $\qquad$ $id_t\Psi_t = H_t\Psi_t$

Interaction destroys product structure.

Question:

- In which regimes: $\Psi_t \approx \prod_{j=1}^{N} \phi_t(x_j)$
- What is ϕ_t?

Mean field for the bosons: The Hartree equation

$$H = \sum_{j=1}^{N} -\Delta_j + \sum_{j=1}^{N} A_t(x_j) + N^{-1} \sum_{k<j} V(x_j - x_k)$$

Interaction „felt“ by each particle of order one

$\Psi_0 = \prod_{j=1}^{N} \phi_0(x_j)$ $\quad\quad$ $i d_t \Psi_t = H_t \Psi_t$

Interaction destroys product structure.

Question:

- In which regimes: $\Psi_t \approx \prod_{j=1}^{N} \phi_t(x_j)$
- What is ϕ_t?

Mean field for the bosons: The Hartree equation

$$H = \sum_{j=1}^{N} -\Delta_j + \sum_{j=1}^{N} A_t(x_j) + N^{-1} \sum_{k<j} V(x_j - x_k)$$

Interaction „felt“ by each particle of order one

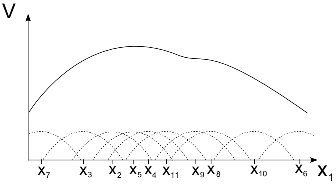
$\Psi_0 = \prod_{j=1}^{N} \phi_0(x_j)$ $\qquad$ $id_t \Psi_t = H_t \Psi_t$

Interaction destroys product structure.

Question:

- In which regimes: $\Psi_t \approx \prod_{j=1}^{N} \phi_t(x_j)$
- What is ϕ_t?

Mean field for "particle 1"

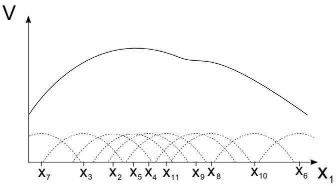


$W(x_1) = N^{-1} \sum_{j=2}^{N} V(x_1 - x_j)$ for fixed, $|\phi_0|^2$- distributed $x_2, \dots, x_N$.
Law of large numbers: $|\phi_0|^2$ close to the empirical density ρ_0.
$W(x_1) \approx V \star |\phi_0|^2(x_1)$ ("Mean field").

Effective Dynamics: Hartree equation

$$i d_t \phi_t = \left(-\Delta + A_t + V \star |\phi_t|^2\right) \phi_t \ .$$

Mean field for "particle 1"

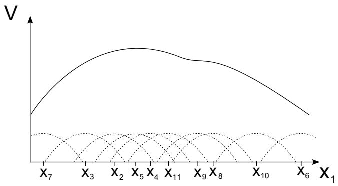


$W(x_1) = N^{-1}\sum_{j=2}^{N} V(x_1 - x_j)$ for fixed, $|\phi_0|^2$- distributed $x_2, \ldots, x_N$.
Law of large numbers: $|\phi_0|^2$ close to the empirical density ρ_0.
$W(x_1) \approx V \star |\phi_0|^2(x_1)$ ("Mean field").

Effective Dynamics: Hartree equation

$$id_t\phi_t = \left(-\Delta + A_t + V \star |\phi_t|^2\right)\phi_t \ .$$

Mean field for "particle 1"

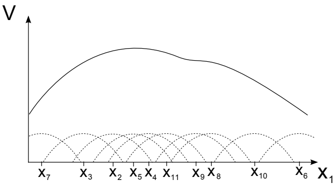


$W(x_1) = N^{-1} \sum_{j=2}^{N} V(x_1 - x_j)$ for fixed, $|\phi_0|^2$- distributed $x_2, \ldots, x_N$.
Law of large numbers: $|\phi_0|^2$ close to the empirical density ρ_0.
$W(x_1) \approx V \star |\phi_0|^2(x_1)$ ("Mean field").

Effective Dynamics: Hartree equation

$$id_t\phi_t = \left(-\Delta + A_t + V \star |\phi_t|^2\right) \phi_t \ .$$

Mean field for "particle 1"

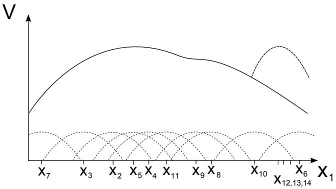


$W(x_1) = N^{-1} \sum_{j=2}^{N} V(x_1 - x_j)$ for fixed, $|\phi_0|^2$- distributed $x_2, \ldots, x_N$.
Law of large numbers: $|\phi_0|^2$ close to the empirical density ρ_0.
$W(x_1) \approx V \star |\phi_0|^2(x_1)$ ("Mean field").

Effective Dynamics: Hartree equation

$$i d_t \phi_t = \left(-\Delta + A_t + V \star |\phi_t|^2\right) \phi_t \ .$$

Grönwall argument

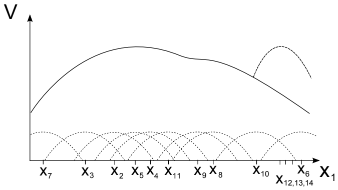


Let α_t be a measure for the dirt in the condensate:

$$d_t\alpha_t \leq C(\alpha_t + o(1))$$

Grönwall: α_t stays small if α_0 was small ($\alpha_t \leq e^{Ct}\alpha_0 + o(1)$)

Grönwall argument

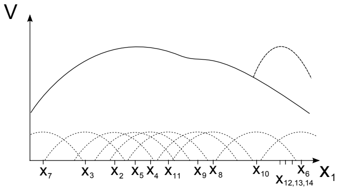


Let α_t be a measure for the dirt in the condensate:

$$d_t\alpha_t \leq C(\alpha_t + o(1))$$

Grönwall: α_t stays small if α_0 was small ($\alpha_t \leq e^{Ct}\alpha_0 + o(1)$)

Grönwall argument



Let α_t be a measure for the dirt in the condensate:

$$d_t\alpha_t \leq C(\alpha_t + o(1))$$

Grönwall: α_t stays small if α_0 was small ($\alpha_t \leq e^{Ct}\alpha_0 + o(1)$)

Some remarks

- Macroscopic equations make only sense for systems with many bosons or heavy, well localized bosons:
- Microscopic system is linear, linearity is broken by the initial condition (product state).
- Flux and density have to be empirical flux and density.
- Good argument takes care of this, looking at Heisenberg-equations is not enough.

Some remarks

- Macroscopic equations make only sense for systems with many bosons or heavy, well localized bosons:
- Microscopic system is linear, linearity is broken by the initial condition (product state).
- Flux and density have to be empirical flux and density.
- Good argument takes care of this, looking at Heisenberg-equations is not enough.

Some remarks

- Macroscopic equations make only sense for systems with many bosons or heavy, well localized bosons:
- Microscopic system is linear, linearity is broken by the initial condition (product state).
- Flux and density have to be empirical flux and density.
- Good argument takes care of this, looking at Heisenberg-equations is not enough.

Some remarks

- Macroscopic equations make only sense for systems with many bosons or heavy, well localized bosons:
- Microscopic system is linear, linearity is broken by the initial condition (product state).
- Flux and density have to be empirical flux and density.
- Good argument takes care of this, looking at Heisenberg-equations is not enough.

Some remarks

- Macroscopic equations make only sense for systems with many bosons or heavy, well localized bosons:
- Microscopic system is linear, linearity is broken by the initial condition (product state).
- Flux and density have to be empirical flux and density.
- Good argument takes care of this, looking at Heisenberg-equations is not enough.

The microscopic system

$$i\partial_t \Psi_N(t) = H_m^N \Psi_N(t), \quad \Psi_N(0) = \Psi_{N0},$$

Pauli-Fierz Hamiltonian

$$H_m^N = \sum_{j=1}^{N} \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}} \right)^2 + \frac{1}{N} \sum_{1 \le j < k \le N} v(x_j - x_k) + H_f$$

second quantized A-field

$$\hat{\boldsymbol{A}}_\kappa(x) = \sum_{\lambda=1,2} \int d^3k \ \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \epsilon_\lambda(k) \left(e^{ikx} a(k,\lambda) + e^{-ikx} a^*(k,\lambda) \right)$$

The microscopic system

$$i\partial_t \Psi_N(t) = H_m^N \Psi_N(t), \quad \Psi_N(0) = \Psi_{N0},$$

Pauli-Fierz Hamiltonian

$$H_m^N = \sum_{j=1}^{N} \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}} \right)^2 + \frac{1}{N} \sum_{1\le j<k\le N} v(x_j - x_k) + H_f$$

second quantized A-field

$$\hat{\boldsymbol{A}}_\kappa(x) = \sum_{\lambda=1,2} \int d^3k \; \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \epsilon_\lambda(k) \left(e^{ikx} a(k,\lambda) + e^{-ikx} a^*(k,\lambda) \right)$$

The microscopic system

$$i\partial_t \Psi_N(t) = H_m^N \Psi_N(t), \quad \Psi_N(0) = \Psi_{N0},$$

Pauli-Fierz Hamiltonian

$$H_m^N = \sum_{j=1}^{N} \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}} \right)^2 + \frac{1}{N} \sum_{1 \le j < k \le N} v(x_j - x_k) + H_f$$

second quantized A-field

$$\hat{\boldsymbol{A}}_\kappa(x) = \sum_{\lambda=1,2} \int d^3k \; \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \epsilon_\lambda(k) \left(e^{ikx} a(k,\lambda) + e^{-ikx} a^*(k,\lambda) \right)$$

The macroscopic system

Hertree-Maxwells equation

$$
\begin{aligned}
i\partial_t\varphi_t(x) &= \Big((-i\nabla - (\kappa\star\boldsymbol{A})(x,t))^2 + (v\star|\varphi_t|^2)(x)\Big)\,\varphi_t(x),\\
\nabla\cdot\boldsymbol{A}(x,t) &= 0,\\
\partial_t\boldsymbol{A}(x,t) &= -\boldsymbol{E}(x,t),\\
\partial_t\boldsymbol{E}(x,t) &= (-\Delta\boldsymbol{A})\,(x,t) - \big(1-\nabla\mathrm{div}\Delta^{-1}\big)\,(\kappa\star\boldsymbol{j}_t)\,(x),\\
\boldsymbol{j}_t(x) &= 2\,\big(\Im(\varphi_t^*\nabla\varphi_t)(x) - |\varphi_t|^2(x)(\kappa\star\boldsymbol{A})(x,t)\big)
\end{aligned}
$$

Grönwall-type estimate

$$\beta^a := \langle\!\langle \Psi_{Nt}, q_1, \Psi_{Nt} \rangle\!\rangle$$

$$\beta^b := \sum_{\lambda=1,2} \int d^3k |k| \langle\!\langle \Psi_{Nt}, \left(\frac{a^*(k,\lambda)}{\sqrt{N}} - \alpha_t^*(k,\lambda) \right) \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda) \right) \Psi_{Nt} \rangle\!\rangle$$

$$\beta^c := \langle\!\langle \left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_t, \alpha_t] \right) \Psi_{Nt}, \left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_t, \alpha_t] \right) \Psi_{Nt} \rangle\!\rangle$$

$$|k|^{1/2} \alpha_t(k,\lambda) := \frac{1}{\sqrt{2}} \epsilon_\lambda(k) \cdot (|k| \mathcal{FT}[\boldsymbol{A}](k,t) - i \mathcal{FT}[\boldsymbol{E}](k,t))$$

β^a measure for the dirt in the condensate
β^b measures the distance of the photon field from a coherent state.
β^c measures the distance of the energies.

Grönwall: $\dot{\beta}_t \leq C(\beta + o(N))$ and β_0 small implies β_t is small.

Grönwall-type estimate

$$\beta^a := \langle\langle \Psi_{Nt}, q_1, \Psi_{Nt} \rangle\rangle$$

$$\beta^b := \sum_{\lambda=1,2} \int d^3k |k| \langle\langle \Psi_{Nt}, \left(\frac{a^*(k,\lambda)}{\sqrt{N}} - \alpha_t^*(k,\lambda) \right) \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda) \right) \Psi_{Nt} \rangle\rangle$$

$$\beta^c := \langle\langle \left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_t, \alpha_t] \right) \Psi_{Nt}, \left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_t, \alpha_t] \right) \Psi_{Nt} \rangle\rangle$$

$$|k|^{1/2} \alpha_t(k,\lambda) := \frac{1}{\sqrt{2}} \epsilon_\lambda(k) \cdot (|k| \mathcal{FT}[\boldsymbol{A}](k,t) - i \mathcal{FT}[\boldsymbol{E}](k,t))$$

β^a measure for the dirt in the condensate

β^b measures the distance of the photon field from a coherent state.

β^c measures the distance of the energies.

Grönwall: $\dot{\beta}_t \leq C(\beta + o(N))$ and β_0 small implies β_t is small.

Grönwall-type estimate

$$\beta^a := \langle\!\langle \Psi_{Nt}, q_1, \Psi_{Nt} \rangle\!\rangle$$

$$\beta^b := \sum_{\lambda=1,2} \int d^3k |k| \langle\!\langle \Psi_{Nt}, \left(\frac{a^*(k,\lambda)}{\sqrt{N}} - \alpha_t^*(k,\lambda) \right) \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda) \right) \Psi_{Nt} \rangle\!\rangle$$

$$\beta^c := \langle\!\langle \left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_t, \alpha_t] \right) \Psi_{Nt}, \left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_t, \alpha_t] \right) \Psi_{Nt} \rangle\!\rangle$$

$$|k|^{1/2} \alpha_t(k,\lambda) := \frac{1}{\sqrt{2}} \epsilon_\lambda(k) \cdot (|k| \mathcal{FT}[\boldsymbol{A}](k,t) - i \mathcal{FT}[\boldsymbol{E}](k,t))$$

β^a measure for the dirt in the condensate
β^b measures the distance of the photon field from a coherent state.
β^c measures the distance of the energies.

Grönwall: $\dot{\beta}_t \leq C(\beta + o(N))$ and β_0 small implies β_t is small.

Grönwall-type estimate

$$\beta^a := \langle\langle \Psi_{Nt}, q_1, \Psi_{Nt} \rangle\rangle$$

$$\beta^b := \sum_{\lambda=1,2} \int d^3k |k| \langle\langle \Psi_{Nt}, \left(\frac{a^*(k,\lambda)}{\sqrt{N}} - \alpha_t^*(k,\lambda) \right) \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda) \right) \Psi_{Nt} \rangle\rangle$$

$$\beta^c := \langle\langle \left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_t, \alpha_t] \right) \Psi_{Nt}, \left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_t, \alpha_t] \right) \Psi_{Nt} \rangle\rangle$$

$$|k|^{1/2} \alpha_t(k,\lambda) := \frac{1}{\sqrt{2}} \epsilon_\lambda(k) \cdot (|k| \mathcal{FT}[\boldsymbol{A}](k,t) - i\mathcal{FT}[\boldsymbol{E}](k,t))$$

β^a measure for the dirt in the condensate
β^b measures the distance of the photon field from a coherent state.
β^c measures the distance of the energies.

Grönwall: $\dot{\beta}_t \leq C(\beta + o(N))$ and β_0 small implies β_t is small.

Grönwall-type estimate

$$\beta^a := \langle\langle \Psi_{Nt}, q_1, \Psi_{Nt} \rangle\rangle$$

$$\beta^b := \sum_{\lambda=1,2} \int d^3k |k| \langle\langle \Psi_{Nt}, \left(\frac{a^*(k,\lambda)}{\sqrt{N}} - \alpha_t^*(k,\lambda) \right) \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda) \right) \Psi_{Nt} \rangle\rangle$$

$$\beta^c := \langle\langle \left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_t, \alpha_t] \right) \Psi_{Nt}, \left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_t, \alpha_t] \right) \Psi_{Nt} \rangle\rangle$$

$$|k|^{1/2} \alpha_t(k,\lambda) := \frac{1}{\sqrt{2}} \epsilon_\lambda(k) \cdot (|k| \mathcal{FT}[\boldsymbol{A}](k,t) - i \mathcal{FT}[\boldsymbol{E}](k,t))$$

β^a measure for the dirt in the condensate
β^b measures the distance of the photon field from a coherent state.
β^c measures the distance of the energies.

Grönwall: $\dot{\beta}_t \leq C(\beta + o(N))$ and β_0 small implies β_t is small.

Grönwall-type estimate

$$\beta^a := \langle\!\langle \Psi_{Nt}, q_1, \Psi_{Nt} \rangle\!\rangle$$

$$\beta^b := \sum_{\lambda=1,2} \int d^3k |k| \langle\!\langle \Psi_{Nt}, \left(\frac{a^*(k,\lambda)}{\sqrt{N}} - \alpha_t^*(k,\lambda) \right) \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda) \right) \Psi_{Nt} \rangle\!\rangle$$

$$\beta^c := \langle\!\langle \left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_t, \alpha_t] \right) \Psi_{Nt}, \left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_t, \alpha_t] \right) \Psi_{Nt} \rangle\!\rangle$$

$$|k|^{1/2} \alpha_t(k,\lambda) := \frac{1}{\sqrt{2}} \epsilon_\lambda(k) \cdot (|k| \mathcal{FT}[\boldsymbol{A}](k,t) - i \mathcal{FT}[\boldsymbol{E}](k,t))$$

β^a measure for the dirt in the condensate
β^b measures the distance of the photon field from a coherent state.
β^c measures the distance of the energies.

Grönwall: $\dot{\beta}_t \leq C(\beta + o(N))$ and β_0 small implies β_t is small.

Thank you!