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Stability of Frustration-Free Ground States of Lattice Fermion Systems

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Joint work with

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Outline

- Fermion lattice systems, interactions, dynamics
- Lieb-Robinson bounds, fermionic conditional expectation
- Gapped ground state phases
- The spectral flow a.k.a. quasi-adiabatic evolution
- Stability of the spectral gap
- Outlook

Fermion Lattice Systems

Spinless fermions on a lattice Γ (a countable set with metric d) are described by the CAR algebra $\mathcal{A}_\Gamma = \mathrm{CAR}(\ell^2(\Gamma))$, generated by creation and annihilation operators a_x^+, a_x, $x \in \Gamma$, which satisfy the Canonical Anticommutation Relations:

$$\{a_x, a_y\} = \{a_x^+, a_y^+\} = 0, \quad \{a_x^+, a_y\} = \delta_{x,y}\mathbb{1}, \quad x, y \in \Gamma.$$

Spin and/or band indices can be included by extending Γ, e.g., by considering $\tilde{\Gamma} = \Gamma \times \{1, \ldots, n\}$. Let $\mathcal{P}_0(\Gamma)$ denote the collection of finite subsets of Γ.
For $X \subset \Gamma$, $\mathcal{A}_X = \mathrm{CAR}(\ell^2(X))$ is naturally identified with the subalgebra of $\mathcal{A}_\Gamma$ generated the a_x, a_x^+, with $x \in X$.

Let $\mathcal{A}_X^+$ and $\mathcal{A}_X^-$ denote the subspaces spanned by the even and odd monomials in a_x, a_x^+, $x \in X$. $\mathcal{A}_\Lambda^+$ is a subalgebra of $\mathcal{A}_\Lambda$, but $\mathcal{A}_\Lambda^-$ is not. Note that if $X \cap Y = \emptyset$, we have

$$AB = BA, \text{ for all } A \in \mathcal{A}_X^+, B \in \mathcal{A}_Y.$$

An interaction Φ for a fermion system on Γ is defined as a map $\mathcal{P}_0(\Gamma) \to \mathcal{A}_\Gamma^+$ such that $\Phi(X) = \Phi(X)^* \in \mathcal{A}_X^+$. For finite Λ, we define the Hamiltonian

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X).$$

Note that we only allow interactions terms that preserve the fermion number parity.

For finite $\Lambda \subset \Gamma$, the Heisenberg dynamics is defined in the usual way

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}, \quad A \in \mathcal{A}_\Lambda.$$

If Φ is not too long-range, in the same way as for spin systems, the thermodynamic limit

$$\lim_{\Lambda \to \Gamma} \tau_t^\Lambda(A) = \tau_t(A), \quad A \in \mathcal{A}_\Gamma^{\mathrm{loc}} = \bigcup_{\Lambda \in \mathcal{P}_0(\Lambda)} \mathcal{A}_\Lambda,$$

defines a strongly continuous one-parameter group of automorphisms on $\mathcal{A}_\Gamma = \overline{\mathcal{A}_\Gamma^{\mathrm{loc}}}$. A standard way to show this is using Lieb-Robinson bounds for interactions with a finite F-norm:

$$\|\Phi\|_F = \sup_{x,y \in \Gamma} F(d(x,y))^{-1} \sum_{\substack{X \in \mathcal{P}_0(\Gamma) \\ x,y \in X}} \|\Phi(X)\|,$$

for a decreasing positive function $F \in L^1(\mathbb{R}^+)$, such that $\sum_{z \in \Gamma} F(d(x,z))F(d(z,y)) \leq F(d(x,y))$, $x, y \in \Gamma$. We can also include time-dependent interactions. For simplicity, we assume that the time-dependence of all interactions is continuous in the operator norm.

In the time-dependent case, the role of $v|t|$, where v is the Lieb-Robinson velocity, is played by the quantity

$$r_{s,t}(\Phi, F) = 2\int_{\min(s,t)}^{\max(s,t)} \|\Phi(\cdot, r)\|_F \, dr.$$

Theorem (Lieb-Robinson Bound for Fermions)

Let Φ be a time-dependent even interaction $\mathcal{P}_0(\Gamma) \to \mathcal{A}_\Gamma^{\mathrm{loc}}$. Let $X, Y \in \mathcal{P}_0(\Gamma)$ with $X \cap Y = \emptyset$. Then, for any $\Lambda \in \mathcal{P}_0(\Gamma)$ with $X \cup Y \subset \Lambda$ and any $A \in \mathcal{A}_X^+$ and $B \in \mathcal{A}_Y$, we have

$$\left\| \left[\tau_{t,s}^{\Lambda}(A), B \right] \right\| \leq 2\|A\|\|B\| \left(e^{r_{s,t}(\Phi,F)} - 1 \right) D(X, Y)$$

for all $t, s \in \mathbb{R}$. Here the quantity $D(X, Y)$ is given by

$$D(X, Y) = \min \left\{ \sum_{x \in X} \sum_{y \in \partial_\Phi Y} F(d(x,y)), \sum_{x \in \partial_\Phi X} \sum_{y \in Y} F(d(x,y)) \right\}$$

The Φ-boundary of X is defined by

$$\partial_\Phi(X) = \{x \in X \mid \text{exist } Z, \Phi(Z) \neq 0, x \in Z, Z \cap X^c \neq \emptyset\}$$

Remark: one can also estimate $\left\| \{\tau^\Lambda_{t,s}(A), B\} \right\|$, for $A \in \mathcal{A}^-_X$ and $B \in \mathcal{A}^-_Y$.
Lieb-Robinson 1972, N-Sims 2006, Hastings-Koma 2006, N-Sims-Ogata 2006, ... , Bru-Pedra 2016, N-Sims-Young in prep.

Conditional Expectation for Fermions

Lieb-Robinson Bounds express the approximate locality of the dynamics: time-evolved local observables are approximately local.

In order to express this quantitatively we need maps $\mathbb{E}_X^\Lambda : \mathcal{A}_\Lambda \to \mathcal{A}_X$, $X \subset \Lambda \in \mathcal{P}_0(\Gamma)$, with the properties of a conditional expectation.

For each $x \in \Gamma$, define

$$u_x^{(0)} = \mathbb{1},\; u_x^{(1)} = a_x^+ + a_x,\; u_x^{(2)} = a_x^+ - a_x,\; u_x^{(3)} = \mathbb{1} - 2a_x^+ a_x.$$

It follows from the CAR that these are unitaries. Clearly, $u_x^{(0)}, u_x^{(3)} \in \mathcal{A}_{\{x\}}^+$, and $u_x^{(1)}, u_x^{(2)} \in \mathcal{A}_{\{x\}}^-$. Therefore, $u_x^{(0)}$ and $u_x^{(3)}$ commute with the elements of $\mathcal{A}_{\Gamma\setminus\{x\}}$, and $u_x^{(1)}$ and $u_x^{(2)}$ commute with $\mathcal{A}_{\Gamma\setminus\{x\}}^+$ and anticommute with $\mathcal{A}_{\Gamma\setminus\{x\}}^-$.

Fix a finite $\Lambda \subset \Gamma$ and X a proper subset of Λ. Fix an ordering of the sites: $\Lambda \setminus X = \{x_1, \ldots, x_k\}$. Then, for each $\alpha_1, \ldots, \alpha_k \in \{0, 1, 2, 3\}$ define the unitary element $u(\alpha) \in \mathcal{A}_\Lambda$ by

$$u(\alpha) = u_{x_1}^{(\alpha_1)} \cdots u_{x_k}^{(\alpha_k)}.$$

The following expression defines a unity-preserving completely positive map $\mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$:

$$\mathbb{E}_X^\Lambda(A) = \frac{1}{4^k} \sum_{\alpha \in \{0,1,2,3\}^k} u(\alpha)^* A u(\alpha), \quad A \in \mathcal{A}_\Lambda. \qquad (1)$$

The map $\mathbb{E}_X^\Lambda$ does not depend on the chosen ordering of the sites. Since the unitaries $u_x^{(k)}, u_y^{(l)}$, $x \neq y$, $k, l \in \{0, 1, 2, 3\}$, either commute or anticommute, any reordering $\tilde{u}(\alpha)$ of $u(\alpha)$ equals either $u(\alpha)$ or $-u(\alpha)$. Either way, the α-term in (1) is not affected.

For $\Lambda \subset \Lambda'$, we have $\mathbb{E}_X^{\Lambda'} \restriction_{\mathcal{A}_\Lambda} = \mathbb{E}_X^{\Lambda}$. Therefore, we can unambiguously define $\mathbb{E}_X : \mathcal{A}_\Gamma^{\rm loc} \to \mathcal{A}_X$ and extend by continuity to $\mathcal{A}_\Gamma$.

Lemma

For all $X \in \mathcal{P}_0(\Gamma)$, the map $\mathbb{E}_X : \mathcal{A}_\Gamma \to \mathcal{A}_\Gamma$ is a unity preserving completely positive map with the following properties:
(i) For $A, B \in \mathcal{A}_X^+, C \in \mathcal{A}_\Gamma$, we have

$$\mathbb{E}_X(ACB) = A\mathbb{E}_X(C)B. \tag{2}$$

(ii) $\mathrm{ran}\mathbb{E}_\Lambda \subset \mathcal{A}_X^+$ and $\mathcal{A}_\Gamma^- \subset \mathrm{ker}\mathbb{E}_X$.
(iii) For $A \in \mathcal{A}_{X^c}$, we have

$$\mathbb{E}_X(A) = \omega^{1/2}(A)\mathbb{1}, \tag{3}$$

where $\omega^{1/2}$ is the quasi-free state of maximal emtropy.

Lemma

Let $X \in \mathcal{P}_0(\Gamma)$, $\epsilon \geq 0$, *and* $A \in \mathcal{A}_\Gamma^+$, *such that for all* $B \in \mathcal{A}_{\Gamma \setminus X}$

$$\|[A, B]\| \leq \epsilon \|B\|.$$

Then

$$\|A - \mathbb{E}_X(A)\| \leq \epsilon.$$

In applications, the ϵ is provided by Lieb-Robinson bounds and A is a time evolved local observable:

$$A = \tau_t(A_0), \quad A_0 \in \mathcal{A}_{X_0}.$$

and

$$\epsilon = 2\|A\| e^{v|t|} |X_0| F(d(X_0, \Gamma \setminus X)).$$

Gapped ground state phases

Our main motivation is to study gapped ground state phases for fermion lattice systems, including topologically ordered phases.

The term gapped refers to the existence of a positive lower bound for the energy of excited states with respect to a ground state, uniformly in the size of the system.

The term phase refers to regions in a interaction space where the gap is positive (open). Phase transitions in interaction space can occur when the gap vanishes (closes).

Topological Order and Discrete Symmetry Breaking are often accompanied by a non-vanishing spectral gap.

The first problem to address is the stability of the spectral gap itself.

Spectral Flow and Automorphic Equivalence

Let $\Phi_s, 0, \leq s \leq 1$, be a differentiable family of short-range interactions, i.e., assume that for some $a, M > 0$, the interactions Φ_s satisfy

$$\sup_{x,y\in\Gamma} e^{ad(x,y)} \sum_{\substack{X\subset\Gamma \\ x,y\in X}} \|\Phi_s(X)\| + |X|\|\partial_s\Phi_s(X)\| \leq M.$$

E.g,

$$\Phi_s = \Phi_0 + s\Psi$$

with both Φ_0 and Ψ finite-range and uniformly bounded.
Let $\Lambda_n \subset \Gamma$, be a sequence of finite volumes, satisfying suitable regularity conditions and suppose that the spectral gap above the ground state (or a low-energy interval) of

$$H_{\Lambda_n}(s) = \sum_{X\subset\Lambda_n} \Phi_s(X)$$

is uniformly bounded below by $\gamma > 0$.

Let $\mathcal{S}(s)$ be the set of thermodynamic limits of ground states of $H_{\Lambda_n}(s)$. E.g., if there is only one ground state, this set contains the state obtained by taking the limit of the infinite lattice: for each observable A,

$$\omega(A) = \lim_{\Lambda_n \to \Gamma} \langle \psi_{\Lambda_n} \mid A\psi_{\Lambda_n} \rangle$$

Theorem (Bachmann-Michalakis-N-Sims 2012)

Under the assumptions of above, there exist automorphisms α_s of the algebra of observables such that $\mathcal{S}(s) = \mathcal{S}_0 \circ \alpha_s$, for $s \in [0,1]$.

The automorphisms α_s can be constructed as the thermodynamic limit of the s-dependent "time" evolution for an interaction $\Omega(X, s)$, which decays almost exponentially.

Concretely, the action of the quasi-local automophisms α_s on observables is given by

$$\alpha_s(A) = \lim_{n\to\infty} V_n^*(s) A V_n(s)$$

where $V_n(s) \in \mathcal{A}_{\Lambda_n}$ is unitary solution of a Schrödinger equation:

$$\frac{d}{ds} V_n(s) = -i D_n(s) V_n(s), \quad V_n(0) = \mathbb{1},$$

with $D_n(s) = \sum_{X \subset \Lambda_n} \Omega(X, s)$.

The α_s satisfy a Lieb-Robinson bound of the form

$$\|[\alpha_s(A), B]\| \leq \|A\| \|B\| \min(|X|, |Y|)(e^{\tilde{v}s} - 1)F(d(X, Y)),$$

where $A \in \mathcal{A}_X, B \in \mathcal{A}_Y$, $0 < d(X, Y)$ is the distance between X and Y. $F(r)$ can be chosen of the form

$$F(r) = Ce^{-\frac{2}{7}\frac{br}{(\log br)^2}}.$$

with $b \sim \gamma/v$, where γ and v are bounds for the gap and the Lieb-Robinson velocity of the interactions Φ_s, i.e., $b \sim a\gamma M^{-1}$.

$$D_\Lambda(s) = \int_{-\infty}^{\infty} w_\gamma(t) \int_0^t e^{iuH_\Lambda(s)} \left[\frac{d}{ds} H_\Lambda(s)\right] e^{-iuH_\Lambda(s)} du\, dt$$

The projections $\mathbb{E}_X$ are used to express $D_\Lambda(S)$ as a quasi-short-range Hamiltonian:

$$D_n(s) = \sum_{X \subset \Lambda_n} \Omega(X, s).$$

These automorphisms implement what Hastings (2004+) called quasi-adiabatic evolution and play a role in proving stability of the gap, as well as the robustness of important features of gapped ground state pages such as topological order and its consequences. (cfr: Giuliani's talk yesterday).

Stability of the Spectral Gap

Statement of a stability theorem for finite volumes ‘without boundary’.

$$H_\Lambda(\epsilon) = \sum_{X \subset \Lambda} \Phi(X) + \epsilon \Psi(X),$$

where Φ and Ψ are even interactions and

- $\Phi(X) \geq 0$ is finite-range, uniformly bounded, and frustration free (cfr Read's talk tomorrow), and $\|\Psi\|_F < \infty$, F decays exponentially;
- $0 \in \operatorname{spec} H_{B_x(r)}(0) \subset \{0\} \cup (\gamma, \infty)$, $x \in \Lambda, r \geq r_0$, for some $\gamma > 0$;
- ground state(s) of $H_\Lambda(0)$ satisfies LTQO.

The Local Topological Quantum Order (LTQO) property was first introduced by Bravyi, Hastings, and Michalakis.

Let P_X denote the projection onto $\ker H_X(0)$. Then, The unperturbed model satisfies LTQO if there is a $q > 0$, and $\alpha \in (0,1)$, such that for all $r \leq (\mathrm{diam}\,\Lambda)^\alpha$, and all $A \in \mathcal{A}^+_{B_x(r)}$,

$$\|P_{B_x(r+\ell)} A P_{B_x(r+\ell)} - \omega_\Lambda(A) P_{B_x(r+\ell)}\| \leq C\|A\|\ell^{-q},$$

with

$$\omega_\Lambda(A) = \mathrm{Tr}(P_\Lambda A)/\mathrm{Tr}(P_\Lambda).$$

Let $E_\Lambda(\epsilon) = \inf \mathrm{spec}(H_\Lambda(\epsilon))$. The gap of $H_\Lambda(\epsilon)$ is defined taking into account that the perturbation may produce a splitting up to an amount δ_Λ of the zero eigenvalue of $H_\Lambda(0)$, which is in general degenerate:

$$\gamma_\delta(H_\Lambda(\epsilon)) = \sup\{\eta > 0 \mid (\delta, \delta+\eta) \cap \mathrm{spec}(H_\Lambda(\epsilon) - E_\Lambda(\epsilon)\mathbb{1}) = \emptyset\}$$

Theorem (Bravyi-Hastings-Michalakis 2011,Michalakis-Zwolak 2013, N-Sims-Young, in prep.)

Let $H_\Lambda(0)$ *as above, and assume the model satisfies LTQO with a sufifciently large* $q > 0$*. Then, for every* $0 < \gamma_0 < \gamma$*, there exists* $\epsilon_0 > 0$ *such that, if* $|\epsilon| < \epsilon_0$*, for sufficiently large* Λ*, we have*

$$\gamma_{\delta_\Lambda}(H_\Lambda(\epsilon)) \geq \gamma_0, \text{ if } |\epsilon| \leq \epsilon_0,$$

where $\delta_\Lambda \leq C(\operatorname{diam}\Lambda)^{-p}$*, for some* $p > 0$*.*

Outlook

- The same techniques can be used to prove robustness of other properties, such topological order, discrete symmetry breaking, ...
- Some non-frustration free models can be handled by considering them as perturbations of frustration free models.
- We hope to also prove stability of anyons, which describe excitations of topologically non-trivial many-body ground states (work in progress with Cha and Naaijkens), at least for simple models such as Kitaev's Toric Code model.
- Proving a gap, needed as a condition for stability, remains a major challenge for non-commuting Hamiltonians in $d > 1$.