Approximations of the Neumann Laplacian in nonuniformly collapsing strips

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3 Effective operator

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5 Examples
Initial

\[ g(x) \]

- \[ \Delta \]

\[ \Lambda_1 \]

\[ a \]
Collapsing regions

Initial

\[ \varepsilon g(x) \text{ nonuniformly collapsing} \]

\[ -\Delta \]

\[ \Lambda_{\varepsilon} \]

a
We consider “thick regions” given by functions $g : [a, \infty) \to (0, \infty)$ with $g(x) \to \infty$ as $x \to \infty$. Is there an effective operator $S = S(g)$ as $\varepsilon \to 0$?

A more delicate question: Is there a family of uniformly collapsing regions $Q_\varepsilon$ whose effective operator coincides with $S$?

Conditions on $g$:

(c1) $C^2$ function and strictly increasing for large values of $x$;

(c2) $j(x) := \frac{g'(x)}{2g(x)}$ and $j'(x)$ are bounded.
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The region of interest is

$$\Lambda_\varepsilon := \{(x, y) \in \mathbb{R}^2 | 0 < y < \varepsilon g(x), \ x \in [a, \infty)\},$$

and the quadratic form (Neumann Laplacian)

$$m_\varepsilon(v) = \int_{\Lambda_\varepsilon} |\nabla v|^2 dx, \ \text{dom } m_\varepsilon = H^1(\Lambda_\varepsilon).$$

After changes of variables, $m_\varepsilon(v)$ is cast as

$$n_\varepsilon(\varphi) := \int_\mathcal{Q} \left( \left| \varphi' - \frac{g'}{2g} \varphi - y \varphi_y \frac{g'}{g} \right|^2 + \frac{\varphi_y^2}{\varepsilon^2 g^2} \right) dxdy,$$

where $\mathcal{Q} := [a, \infty) \times (0, 1)$ is a fixed region. Note that, as $\varepsilon \to 0$,

$$n_\varepsilon(\varphi) \longrightarrow n(\varphi) := \begin{cases} \int_\mathcal{Q} \left| \varphi' - \frac{g'}{2g} \varphi \right|^2 dxdy, & \text{if } \varphi_y = 0, \\ \infty, & \text{if } \varphi_y \neq 0. \end{cases}$$

Let $S_\varepsilon$ and $S$ be the operators associated with $n_\varepsilon$ and $n$, respectively.
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(Un)Bounded region

Let $\mathcal{L} := \{ \varphi(x, y) = w(x)1 | w \in L^2([a, \infty)) \}$.

Theorem (1) (by Kato-Robinson Theorem)

For all $f \in L^2(Q)$ one has, as $\varepsilon \to 0$,

$$\| S^{-1}_\varepsilon f - (S^{-1} \oplus 0) f \| \to 0,$$

where $0$ is the null operator on $\mathcal{L}^\perp$. 
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where $0$ is the null operator on $\mathcal{L}^\perp$.***
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4 Uniformly collapsing approximations

5 Examples
(Un)Bounded region

The goal now is to characterize $S$: for this we need (c2), i.e., bounded $j = \frac{g'}{2g}$ and $j'$.

Theorem (2)

For $g$ as above, we have

$$(Sw)(x) := -w''(x) + \varrho(x)w(x),$$

with $\varrho(x) := j^2(x) + j'(x)$ and a Robin condition at the end point $a$, that is,

$$\text{dom } S = \{w \in H^2([a, \infty)) \mid j(a)w(a) = w'(a)\}. $$
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Diverging region

Second main goal: finding uniformly collapsing regions $Q_\varepsilon$ whose effective operator coincides with $S$. 

\[
1/\varepsilon^\alpha \quad (0<\alpha<1)
\]

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$Q_\varepsilon$
Pick bounded functions $g_\varepsilon : [a, +\infty) \to \mathbb{R}$ as in the previous figure, which converges pointwise to $g$ with collapsing $\varepsilon g$ (nonuniformly) and $\varepsilon g_\varepsilon$ (uniformly).

Recall that $Q_\varepsilon$ denotes the region below $\varepsilon g_\varepsilon(x)$. Consider the Neumann quadratic form

$$f_\varepsilon(\psi) = \int_{Q_\varepsilon} |\nabla \psi|^2 \, dx \, dy,$$

$$\text{dom } f_\varepsilon = H^1(Q_\varepsilon).$$

Set $Q := [a, \infty) \times (0, 1)$. After changes of variables, we pass to

$$h_\varepsilon(\psi) = \int_Q \left( \left| \psi' - \frac{g'_\varepsilon}{2g_\varepsilon} \psi - y \frac{g'_\varepsilon}{g_\varepsilon} \psi_y \right|^2 + \frac{|\psi_y|^2}{\varepsilon^2 g_\varepsilon^2} \right) \, dx \, dy,$$

$$\text{dom } h_\varepsilon = H^1(Q) \subset L^2(Q).$$

Denote by $H_\varepsilon$ the associated operator whose behavior we are interested in understanding as $\varepsilon \to 0$. 
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Uniformly collapsing approximations

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• First a **reduction of dimension**. Consider again the subspace

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\mathcal{L} = \{ w(x) \mid w \in L^2([a, \infty)) \},
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the one-dimensional quadratic form

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t_\varepsilon(w) := h_\varepsilon(w 1) = \int_a^\infty \left| w' - \frac{g'_\varepsilon}{2g_\varepsilon} w \right|^2 dx,
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and denote by $T_\varepsilon$ the associated operator.

Under the above conditions:

**Theorem (3)(based on Friedlander & Solomyak method)**

*For $g$ as above, one has*

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\left\| H_\varepsilon^{-1} - (T_\varepsilon^{-1} \oplus 0) \right\| \to 0, \quad \varepsilon \to 0,
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*where $0$ is the null operator on the subspace $\mathcal{L}^\perp$.***
Uniformly collapsing regions

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• $T_\epsilon$ is already **unidimensional**. The next task is the limit of $T_\epsilon$.

**Theorem (4)** (based on Bedoya, deO & Verri)

Let $g : [a, \infty) \to \mathbb{R}$ be as above. Then:

(A) The sequence $T_\epsilon$ converges in the strong resolvent sense to $S$.

(B) If $j(x) = \frac{g'(x)}{2g(x)}$ vanishes as $x \to \infty$, then

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\|T_\epsilon^{-1} - S^{-1}\| \to 0.
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Recall: $Sw = -w'' + \varrho(x)w$, with $\varrho = j^2 + j'$, and b.c. $j(a)w(a) = w'(a)$. 
Uniformly collapsing approximations

Uniformly collapsing regions

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Uniformly collapsing regions

In summary:

through such uniformly collapsing $Q_\epsilon$ we have recovered $S$ (initially found from Kato-Robinson) as the effective operator.

Especially in case

$$j(x) = \frac{g'(x)}{2g(x)} \to 0, \quad x \to \infty,$$

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5 Examples
Class I. [Power law] Take $g(x) = \gamma x^\beta$, $\gamma, \beta > 0$, for $x \geq 1$.

Then $a = 1$ and $j(x) = \beta/(2x)$ vanishes at infinity. So, as $\epsilon \to 0$, there is a norm resolvent convergence (in uniformly collapsing regions) to the effective operator

$$(Sw)(x) = -w''(x) + \frac{\beta(\beta - 2)}{4x^2}w(x), \quad \frac{\beta}{2}w(1) = w'(1).$$
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Note that for \( g(x) = \gamma x^\beta \) the effective potential \( \varphi(x) = \frac{\beta(\beta-2)}{4x^2} \):

- does not depend on \( \gamma \);
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- does not depend on $\gamma$;
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- is negative for $0 < \beta < 2$ and positive for $\beta > 2$. 
Class II. [Exponential of a power] For $x \geq 1$, consider $g(x) = \gamma e^{x^\beta}$, $\gamma, \beta > 0$.

Now $j(x) = \frac{\beta^2}{2x^{1-\beta}}$: it is bounded only if $\beta \leq 1$ and vanishes at infinity if $\beta < 1$.

The effective operator in this case is

$$(Sw)(x) = (S^\beta w)(x) := -w''(x) + q^\beta(x)w(x), \quad \beta \frac{2}{2}w(1) = w'(1),$$

with $q^\beta(x) := \frac{1}{4} \left( \frac{\beta^2}{x^{2(1-\beta)}} - \frac{2\beta(1-\beta)}{x^{2-\beta}} \right)$.

By Theorem 4, if $0 < \beta < 1$, one has (in $Q_\varepsilon$) norm resolvent convergence to the effective operator, whereas for $\beta = 1$ we have strong convergence.
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By Theorem 4, if $0 < \beta < 1$, one has (in $Q_\varepsilon$) norm resolvent convergence to the effective operator, whereas for $\beta = 1$ we have strong convergence.
For “regions” $g(x) = \gamma e^{x^\beta}$, the effective potential $\varrho^\beta(x) = \frac{1}{4} \left( \frac{\beta^2}{x^{2(1-\beta)}} - \frac{2\beta(1-\beta)}{x^{2-\beta}} \right)$:

- does not depend on $\gamma$;

- for $0 < \beta < 1$, it is bounded and vanishes at $\infty$. Furthermore, it is negative in a neighborhood of 1 and positive for large values of $x$;

- for $\beta = 1$ (the exponentially thick region), it is constant and equals to $1/4$, and so the transition point from norm to strong resolvent approximations.
For “regions” $g(x) = \gamma e^{x^\beta}$, the effective potential $g^\beta(x) = \frac{1}{4} \left( \frac{\beta^2}{x^{2(1-\beta)}} - \frac{2\beta(1-\beta)}{x^{2-\beta}} \right)$:

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Examples

If time permits.

Final remarks:

- The condition that \( j(x) \) is bounded implies that \( g(x) \leq \gamma e^{\kappa x} \).

In the borderline case \( g(x) = \gamma e^{\kappa x} \) one has the effective potential \( g(x) = \frac{\kappa^2}{4} \).

- For \( g(x) = x^3 + \frac{1}{2} \frac{\sin(x^3)}{x} \), \( x \geq 1 \), it follows that \( j(x) \) vanishes at infinity and \( g(x) \) is bounded but oscillates wildly for large \( x \).

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Thank you.