Approximations of the Neumann Laplacian in nonuniformly collapsing strips

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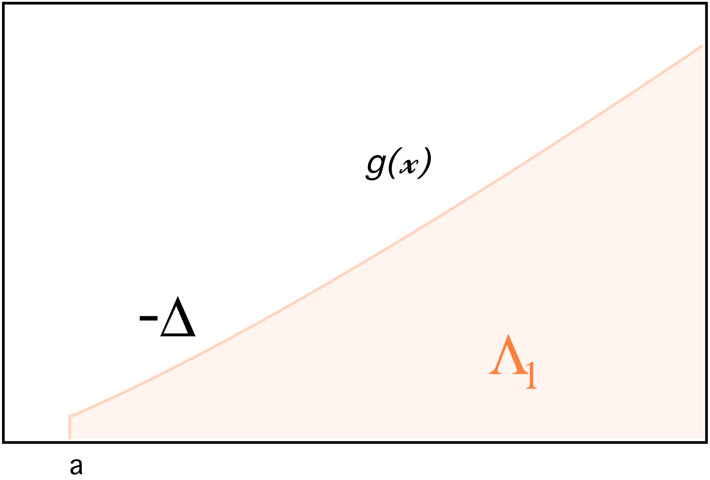
Sources

- R. Bedoya, C. R. de Oliveira & A. A. Verri: Complex Γ-convergence and magnetic Dirichlet Laplacian in bounded thin tubes. J. Spectr. Theory 4 (2014) 621–642
- C. R. de Oliveira & A. A. Verri: On the Neumann Laplacian in nonuniformly collapsing strips. Preprint.
- L. Friedlander & M. Solomyak: On the spectrum of the Dirichlet Laplacian in a narrow infinite strip. Amer. Math. Soc. Transl. 225 (2008) 103–116
- J. K. Hale & G. Raugel: Reaction-diffusion equation in thin domains. J. Math. pures et appl. 71 (1992) 33–95

Initial



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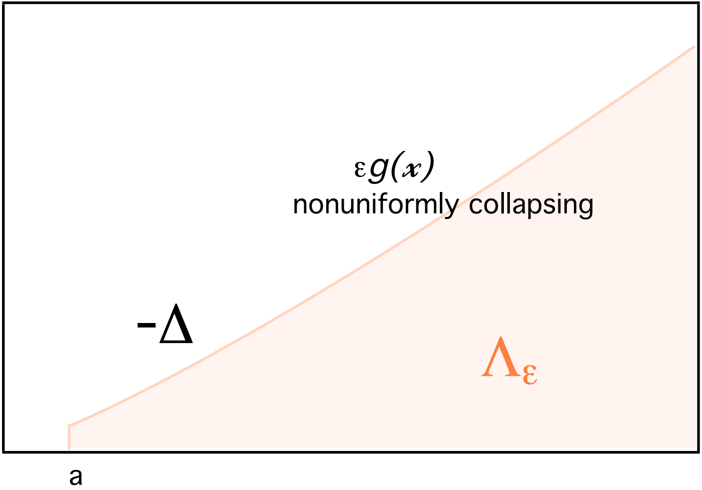

(Un)Bounded region

• We consider "thick regions" given by functions $g : [a, \infty) \to (0, \infty)$ with $g(x) \to \infty$ as $x \to \infty$. Is there an effective operator $S = S(g)$ as $\varepsilon \to 0$?

•• A more delicate question: Is there a family of uniformly collapsing regions Q_ε whose effective operator coincides with S ?

• • • Conditions on g:
(c1) C^2 function and strictly increasing for large values of x;

(c2) $j(x) := \frac{g'(x)}{2g(x)}$ and $j'(x)$ are bounded.

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The region of interest is

$$\Lambda_\varepsilon := \left\{(x,y) \in \mathbb{R}^2 \mid 0 < y < \varepsilon g(x),\ \ x \in [a,\infty)\right\},$$

and the quadratic form (Neumann Laplacian)

$$m_\varepsilon(v) = \int_{\Lambda_\varepsilon} |\nabla v|^2 \mathrm{d}x, \quad \mathrm{dom}\, m_\varepsilon = H^1(\Lambda_\varepsilon).$$

After changes of variables, $m_\varepsilon(v)$ is cast as

$$n_\varepsilon(\varphi) := \int_Q \left(\left| \varphi' - \frac{g'}{2g}\varphi - y\,\varphi_y \frac{g'}{g} \right|^2 + \frac{|\varphi_y|^2}{\varepsilon^2 g^2} \right) \mathrm{d}x\mathrm{d}y,$$

where $Q := [a,\infty) \times (0,1)$ is a fixed region. Note that, as $\varepsilon \to 0$,

$$n_\varepsilon(\varphi) \longrightarrow n(\varphi) := \begin{cases} \int_Q \left|\varphi' - \frac{g'}{2g}\varphi\right|^2 \mathrm{d}x\mathrm{d}y, & \text{if } \ \varphi_y = 0, \\ \infty, \quad \text{if} & \varphi_y \neq 0. \end{cases}$$

Let S_ε and S be the operators associated with n_ε and n, respectively.

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Let $\mathcal{L} := \{\varphi(x,y) = w(x)1 \mid w \in \mathrm{L}^2([a,\infty))\}$.

Theorem (1) (by Kato-Robinson Theorem)

For all $f \in \mathrm{L}^2(Q)$ one has, as $\varepsilon \to 0$,

$$\left\| S_\varepsilon^{-1} f - (S^{-1} \oplus 0) f \right\| \longrightarrow 0\,,$$

where 0 is the null operator on $\mathcal{L}^\perp$.

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The goal now is to characterize S: for this we need (c2), i.e., bounded $j = \frac{g'}{2g}$ and j'.

Theorem (2)

For g as above, we have

$$(Sw)(x) := -w''(x) + \varrho(x)w(x),$$

with $\varrho(x) := j^2(x) + j'(x)$ and a Robin condition at the end point a, that is,

$$\operatorname{dom} S = \{w \in H^2([a,\infty)) \mid j(a)\,w(a) = w'(a)\}.$$

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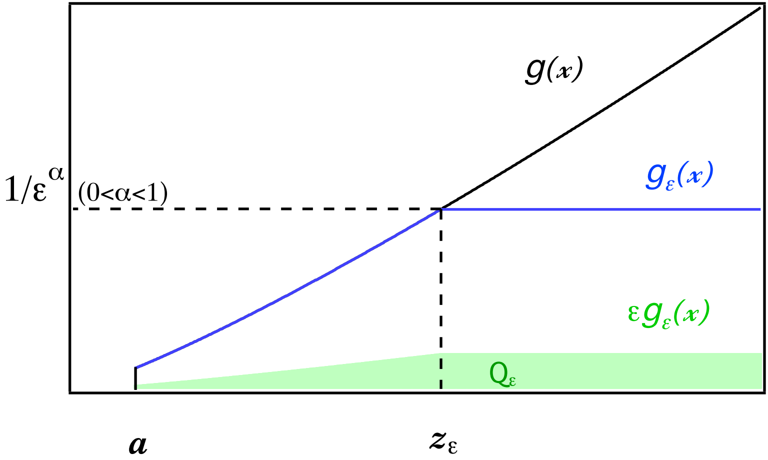
Diverging region

Second main goal: finding uniformly collapsing regions Q_ε whose effective operator coincides with S.



Uniformly collapsing approximations

Pick bounded functions $g_\varepsilon : [a, +\infty) \to \mathbb{R}$ as in the previous figure, which converges pointwise to g with collapsing εg (nonuniformly) and $\varepsilon g_\varepsilon$ (uniformly).

Recall that Q_ε denotes the region below $\varepsilon g_\varepsilon(x)$. Consider the Neumann quadratic form

$$f_\varepsilon(\psi) = \int_{Q_\varepsilon} |\nabla\psi|^2 \, dx dy, \qquad \mathrm{dom}\, f_\varepsilon = H^1(Q_\varepsilon).$$

Set $Q := [a, \infty) \times (0, 1)$. After changes of variables, we pass to

$$h_\varepsilon(\psi) = \int_Q \left(\left| \psi' - \frac{g_\varepsilon'}{2g_\varepsilon}\psi - y\frac{g_\varepsilon'}{g_\varepsilon}\psi_y \right|^2 + \frac{|\psi_y|^2}{\varepsilon^2 g_\varepsilon^2} \right) dx dy, \qquad (1)$$

$\mathrm{dom}\, h_\varepsilon = H^1(Q) \subset L^2(Q)$. Denote by H_ε the associated operator **whose behavior we are interested in understanding as $\varepsilon \to 0$.**

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Uniformly collapsing regions

- First a **reduction of dimension**. Consider again the subspace

$$\mathcal{L} = \{w(x)\,1 \mid w \in \mathrm{L}^2([a,\infty))\} ,$$

the one-dimensional quadratic form

$$t_\varepsilon(w) := h_\varepsilon(w\,1) = \int_a^\infty \left|w' - \frac{g'_\varepsilon}{2g_\varepsilon} w\right|^2 \mathrm{d}x\,, \qquad \mathrm{dom}\, t_\varepsilon = H^1([a,\infty)), \tag{2}$$

and denote by T_ε the associated operator.

Under the above conditions:

Theorem (3)(based on Friedlander & Solomyak method)

For g as above, one has

$$\left\| H_\varepsilon^{-1} - (T_\varepsilon^{-1} \oplus 0) \right\| \longrightarrow 0, \quad \varepsilon \to 0\,,$$

where 0 *is the null operator on the subspace* $\mathcal{L}^\perp$.

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- T_ε is already **unidimensional**. The next task is the limit of T_ε.

Theorem (4)(based on Bedoya, deO & Verri)

Let $g : [a, \infty) \to \mathbb{R}$ be as above. Then:

(A) The sequence T_ε converges in the strong resolvent sense to S.

(B) If $j(x) = \frac{g'(x)}{2g(x)}$ vanishes as $x \to \infty$, then

$$\left\| T_\varepsilon^{-1} - S^{-1} \right\| \longrightarrow 0.$$

Recall: $Sw = -w'' + \varrho(x)w$, with $\varrho = j^2 + j'$, and b.c. $j(a)w(a) = w'(a)$.

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In summary:

through such uniformly collapsing Q_ε we have recovered S (initially found from Kato-Robinson) as the effective operator.

Especially in case

$$j(x) = \frac{g'(x)}{2g(x)} \to 0, \quad x \to \infty,$$

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Examples

Class I. [Power law] Take $g(x) = \gamma x^{\beta}, \quad \gamma, \beta > 0$, for $x \geq 1$.

Then $a = 1$ and $j(x) = \beta/(2x)$ vanishes at infinity. So, as $\varepsilon \to 0$, there is a norm resolvent convergence (in uniformly collapsing regions) to the effective operator

$$(Sw)(x) = -w''(x) + \frac{\beta(\beta-2)}{4x^2} w(x)\,, \qquad \frac{\beta}{2}\, w(1) = w'(1)\,.$$

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- does not depend on γ;
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Now $j(x) = \frac{\beta}{2\, x^{1-\beta}}$: it is bounded only if $\beta \leq 1$ and vanishes at infinity if $\beta < 1$.

The effective operator in this case is

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with $\varrho^\beta(x) := \frac{1}{4}\left(\frac{\beta^2}{x^{2(1-\beta)}} - \frac{2\beta(1-\beta)}{x^{2-\beta}}\right)$.

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By Theorem 4, if $0 < \beta < 1$, one has (in Q_ε) norm resolvent convergence to the effective operator, whereas for $\beta = 1$ we have strong convergence.

Examples

For "regions" $g(x) = \gamma\, e^{x^\beta}$, the effective potential $\varrho^\beta(x) = \frac{1}{4}\left(\frac{\beta^2}{x^{2(1-\beta)}} - \frac{2\beta(1-\beta)}{x^{2-\beta}}\right)$:

- does not depend on γ;
- for $0 < \beta < 1$, it is bounded and vanishes at ∞. Furthermore, it is negative in a neighborhood of 1 and positive for large values of x;
- for $\beta = 1$ (the exponentially thick region), it is constant and equals to $1/4$, and so the transition point from norm to strong resolvent approximations.

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Examples

If time permits.

Final remarks:

• The condition that $j(x)$ is bounded implies that $g(x) \le \gamma\, e^{\kappa x}$.

In the borderline case $g(x) = \gamma\, e^{\kappa x}$ one has the effective potential $\varrho(x) = \frac{\kappa^2}{4}$.

• For $g(x) = x^3 + \frac{1}{2}\frac{\sin(x^3)}{x}$, $x \ge 1$, it follows that $j(x)$ vanishes at infinity and $\varrho(x)$ is bounded but oscillates wildly for large x.

• Naturally, a spectral analysis should be undertaken ...

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Thanks

Thank you.