Local Eigenvalue
Asymptotics of the
Perturbed Krein Laplacian

QMath13
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Based on the preprint:

V. Bruneau, G. Raikov,
*Spectral properties of harmonic Toeplitz operators and applications to the perturbed Krein Laplacian*, arXiv:1609.08229.
1. The Krein Laplacian and its perturbations

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with boundary $\partial \Omega \in C^{\infty}$. For $s \in \mathbb{R}$, we denote by $H^s(\Omega)$ and $H^s(\partial \Omega)$ the Sobolev spaces on $\Omega$ and $\partial \Omega$ respectively, and by $H^s_0(\Omega)$, $s > 1/2$, the closure of $C^\infty_0(\Omega)$ in $H^s(\Omega)$.

Define the minimal Laplacian

$$\Delta_{\text{min}} := \Delta, \quad \text{Dom } \Delta_{\text{min}} = H^2_0(\Omega).$$

Then $\Delta_{\text{min}}$ is symmetric and closed but not self-adjoint in $L^2(\Omega)$ since

$$\Delta_{\text{max}} := \Delta_{\text{min}}^* = \Delta,$$

$$\text{Dom } \Delta_{\text{max}} = \left\{ u \in L^2(\Omega) \mid \Delta u \in L^2(\Omega) \right\}.$$ 

We have

$$\text{Ker } \Delta_{\text{max}} = \mathcal{H}(\Omega) := \left\{ u \in L^2(\Omega) \mid \Delta u = 0 \text{ in } \Omega \right\},$$

$$\text{Dom } \Delta_{\text{max}} = \mathcal{H}(\Omega) + H^2_D(\Omega)$$

where $H^2_D(\Omega) := H^2(\Omega) \cap H^1_0(\Omega)$. 
Introduce the Krein Laplacian

\[ K := -\Delta, \quad \text{Dom } K = \mathcal{H}(\Omega) + H^2_0(\Omega). \]

The operator \( K \geq 0 \), self-adjoint in \( L^2(\Omega) \), is the von Neumann-Krein “soft” extension of \(-\Delta_{\text{min}}\), remarkable for its property that any other self-adjoint extension \( S \geq 0 \) of \(-\Delta_{\text{min}}\) satisfies

\[
(S + I)^{-1} \leq (K + I)^{-1}.
\]

We have \( \text{Ker } K = \mathcal{H}(\Omega) \). Moreover, \( \text{Dom } K \) can be described in terms of the Dirichlet-to-Neumann operator \( \mathcal{D} \). For \( f \in C^\infty(\partial \Omega) \), set

\[
\mathcal{D} f = \frac{\partial u}{\partial \nu}|_{\partial \Omega},
\]

where \( \nu \) is the outer normal unit vector at \( \partial \Omega \), \( u \) is the solution of the boundary-value problem

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
u = f & \text{on } \partial \Omega.
\end{cases}
\]

Thus, \( \mathcal{D} \) is a first-order elliptic \( \Psi \text{DO} \); hence, it extends to a bounded operator form \( H^s(\partial \Omega) \) into \( H^{s-1}(\partial \Omega) \), \( s \in \mathbb{R} \). In particular, \( \mathcal{D} \) with domain \( H^1(\partial \Omega) \) is self-adjoint in \( L^2(\partial \Omega) \).
Then we have

\[
\text{Dom } K = \left\{ \left. u \in \text{Dom } \Delta_{\text{max}} \right| \frac{\partial u}{\partial \nu}|_{\partial \Omega} = D \left( u|_{\partial \Omega} \right) \right\}.
\]

The Krein Laplacian \( K \) arises naturally in the so called \textit{buckling problem}:

\[
\left\{ \begin{array}{l}
\Delta^2 u = -\lambda \Delta u, \\
u|_{\partial \Omega} = \frac{\partial u}{\partial \nu}|_{\partial \Omega} = 0, \\
u \in \text{Dom } \Delta_{\text{max}}.
\end{array} \right.
\]
Let $L$ be the restriction of $K$ onto $\text{Dom } K \cap \mathcal{H}(\Omega)^\perp$ where $\mathcal{H}(\Omega)^\perp := L^2(\Omega) \ominus \mathcal{H}(\Omega)$. Then, $L$ is self-adjoint in $\mathcal{H}(\Omega)^\perp$.

**Proposition 1.** The spectrum of $L$ is purely discrete and positive, and, hence, $L^{-1}$ is compact in $\mathcal{H}(\Omega)^\perp$. As a consequence, $\sigma_{\text{ess}}(K) = \{0\}$, and the zero is an isolated eigenvalue of $K$ of infinite multiplicity.

Let $V \in C(\overline{\Omega}; \mathbb{R})$. Then the operator $K + V$ with domain $\text{Dom } K$ is self-adjoint in $L^2(\Omega)$. In the sequel, we will investigate the spectral properties of $K + V$. 
It should be underlined here that the perturbations $K + V$ are of different nature than the perturbations $K_V$ discussed in the article M. S. Ashbaugh, F. Gesztesy, M. Mitrea, G. Teschl, *Spectral theory for perturbed Krein Laplacians in nonsmooth domains*, Adv. Math. **223** (2010), 1372–1467, where the authors assume that $V \geq 0$, and set

\[ K_{V,\text{max}} := -\Delta + V, \quad \text{Dom} \ K_{V,\text{max}} := \text{Dom} \Delta_{\text{max}}, \]

\[ K_V := -\Delta + V, \quad \text{Dom} \ K_V := \text{Ker} \ K_{V,\text{max}} + H^2_0(\Omega). \]

Thus, if $V \neq 0$, then the operators $K_V$ and $K_0 = K$ are self-adjoint on different domains, while the operators $K + V$ are all self-adjoint on $\text{Dom} \ K$. Moreover, for any $0 \leq V \in C(\bar{\Omega})$, we have $K_V \geq 0$, $\sigma_{\text{ess}}(K_V) = \{0\}$, and the zero is an isolated eigenvalue of $K_V$ of infinite multiplicity. As we will see, the properties of $K + V$ could be quite different.
Theorem 1. Let $V \in C(\overline{\Omega}; \mathbb{R})$. Then we have

$$\sigma_{\text{ess}}(K + V) = V(\partial \Omega).$$

In particular, $\sigma_{\text{ess}}(K + V) = \{0\}$ if and only if $V_{|\partial \Omega} = 0$.

In the rest of the talk, we assume that $0 \leq V \in C(\overline{\Omega})$ with

$$V_{|\partial \Omega} = 0, \quad (1)$$

and will investigate the asymptotic distribution of the discrete spectrum of the operators $K \pm V$, adjoining the origin.

Set $\lambda_0 := \inf \sigma(L)$,

$$\mathcal{N}_-(\lambda) := \text{Tr} \mathbb{1}_{(-\infty,-\lambda)}(K - V), \; \lambda > 0,$$

$$\mathcal{N}_+(\lambda) := \text{Tr} \mathbb{1}_{(\lambda,\lambda_0)}(K + V), \; \lambda \in (0, \lambda_0).$$
Let $P : L^2(\Omega) \to L^2(\Omega)$ be the orthogonal projection onto $\mathcal{H}(\Omega)$. Introduce the harmonic Toeplitz operator

$$T_V := PV : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega).$$

If $V \in C(\overline{\Omega})$, then $T_V$ is compact if and only if (1) holds true.

Let $T = T^*$ be a compact operator in a Hilbert space. Set

$$n(s; T) := \text{Tr} \mathbb{1}_{(s, \infty)}(T), \quad s > 0.$$ 

Thus, $n(s; T)$ is just the number of the eigenvalues of the operator $T$ larger than $s$, counted with their multiplicities.
Theorem 2. Assume that $0 \leq V \in C(\overline{\Omega})$ and $V|_{\partial\Omega} = 0$. Then for any $\varepsilon \in (0, 1)$ we have

$$n(\lambda; T_V) \leq \mathcal{N}_-(\lambda) \leq n((1 - \varepsilon)\lambda; T_V) + O(1),$$

and

$$n((1 + \varepsilon)\lambda; T_V) + O(1) \leq \mathcal{N}_+(\lambda) \leq n((1 - \varepsilon)\lambda; T_V) + O(1),$$

as $\lambda \downarrow 0$.

The proof of Theorem 2 is based on suitable versions of the Birman–Schwinger principle.
2. Spectral asymptotics of $T_V$ for $V$ of power-like decay at $\partial \Omega$

Let $a, \tau \in C^\infty(\overline{\Omega})$ satisfy $a > 0$ on $\overline{\Omega}$, $\tau > 0$ on $\Omega$, and $\tau(x) = \text{dist}(x, \partial \Omega)$ for $x$ in a neighborhood of $\partial \Omega$. Assume

$$V(x) = \tau(x)^\gamma a(x), \quad \gamma \geq 0, \quad x \in \Omega. \quad (2)$$

Set $a_0 := a|_{\partial \Omega}$.

**Theorem 3.** Assume that $V$ satisfies (2) with $\gamma > 0$. Then we have

$$n(\lambda; T_V) = C \lambda^{-\frac{d-1}{\gamma}} \left(1 + O(\lambda^{1/\gamma})\right), \quad \lambda \downarrow 0,$$

(3)

where

$$C := \omega_{d-1} \left(\frac{\Gamma(\gamma + 1)^{1/\gamma}}{4\pi}\right)^{d-1} \int_{\partial \Omega} a_0(y)^{\frac{d-1}{\gamma}} dS(y),$$

(4)

and $\omega_n = \pi^{n/2}/\Gamma(1 + n/2)$ is the volume of the unit ball in $\mathbb{R}^n$, $n \geq 1$. 

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Idea of the proof of Theorem 3:

Assume that $f \in L^2(\partial \Omega)$, $s \in \mathbb{R}$. Then the boundary-value problem

$$
\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
u = f & \text{on } \partial \Omega,
\end{cases}
$$

admits a unique solution $u \in H^{1/2}(\Omega)$, and the mapping $f \mapsto u$ defines an isomorphism between $L^2(\partial \Omega)$ and $H^{1/2}(\Omega)$. Set

$$u := Gf.$$

The operator $G : L^2(\partial \Omega) \to L^2(\Omega)$ is compact, and

$$\text{Ker } G = \{0\}, \quad \overline{\text{Ran } G} = \mathcal{H}(\Omega).$$

Set $J := G^*G$. Then the operator $J = J^* \geq 0$ is compact in $L^2(\partial \Omega)$, and $\text{Ker } J = \{0\}$. Hence, the operator $J^{-1}$ is well defined as an unbounded positive operator, self-adjoint in $L^2(\partial \Omega)$. 
Let
\[ G = U|G| = UJ^{1/2} \]
be the polar decomposition of the operator \( G \), where \( U : L^2(\partial \Omega) \to L^2(\Omega) \) is an isometric operator with \( \text{Ker} \ U = \{0\} \) and \( \text{Ran} \ U = \mathcal{H}(\Omega) \).

**Proposition 2.** The orthogonal projection \( P \) onto \( \mathcal{H}(\Omega) \) satisfies
\[ P = GJ^{-1}G^* = UU^*. \]

Assume that \( V \) satisfies (2) with \( \gamma \geq 0 \), and set \( J_V := G^*VG \).

**Proposition 3.** Let \( V \) satisfy (2) with \( \gamma > 0 \). Then the operator \( T_V \) is unitarily equivalent to the operator \( J^{-1/2}J_VJ^{-1/2} \).

**Proof.** We have
\[ PVP = UJ^{-1/2}J_VJ^{-1/2}U^*, \]
and \( U \) maps unitarily \( L^2(\partial \Omega) \) onto \( \mathcal{H}(\Omega) \). \( \square \)
**Proposition 4.** Under the assumptions of Proposition 3 the operator \( J^{-1/2} J_V J^{-1/2} \) is a \( \Psi \text{DO} \) with principal symbol

\[
2^{-\gamma} \Gamma(\gamma + 1) |\eta|^{-\gamma} a_0(y), \quad (y, \eta) \in T^*\partial\Omega.
\]

The proof of Proposition 4 is based on the pseudo-differential calculus due to L. Boutet de Monvel.

Further, under the assumptions of Theorem 3, we have \( \text{Ker} \ J^{-1/2} J_V J^{-1/2} = \{0\} \). Define the operator

\[
A := \left( J^{-1/2} J_V J^{-1/2} \right)^{-1/\gamma}.
\]

Then \( A \) is a \( \Psi \text{DO} \) with principal symbol

\[
2\Gamma(\gamma + 1)^{-1/\gamma} |\eta| a_0(y)^{-1/\gamma}, \quad (y, \eta) \in T^*\partial\Omega.
\]
By Proposition 3 and the spectral theorem, we have

\[ n(\lambda; T_V) = \text{Tr} \mathbb{1}_{(-\infty, \lambda^{-1/\gamma})}(A), \quad \lambda > 0. \quad (5) \]


\[ \text{Tr} \mathbb{1}_{(-\infty, E)}(A) = C E^{d-1} (1 + O(E^{-1})), \quad E \to \infty, \quad (6) \]

the constant \( C \) being defined in (4). Combining (5) and (6), we arrive at (3).
3. Spectral asymptotics of $T_V$ for radially symmetric compactly supported $V$

In this section we discuss the eigenvalue asymptotics of $T_V$ in the case where $\Omega$ is the unit ball in $\mathbb{R}^d$, $d \geq 2$, while $V$ is compactly supported in $\Omega$, and possesses a partial radial symmetry.

Set

$$B_r := \{ x \in \mathbb{R}^d \mid |x| < r \}, \quad d \geq 2, \quad r \in (0, \infty).$$

**Proposition 5.** Let $\Omega = B_1$. Assume that $0 \leq V \in C(\overline{B_1})$, and $\text{supp } V = \overline{B_c}$ for some $c \in (0, 1)$. Suppose moreover that for any $\delta \in (0, c)$ we have $\inf_{x \in B_\delta} V(x) > 0$. Then

$$\lim_{\lambda \downarrow 0} |\ln \lambda|^{-d+1} n(\lambda; T_V) = \frac{2^{-d+2}}{(d-1)!|\ln c|^{d-1}}.$$
The proof of Proposition 5 is based on the following

**Lemma 1.** Let $\Omega = B_1$, $V = b \mathbb{1}_{B_c}$ with some $b > 0$ and $c \in (0, 1)$. Then we have

$$n(\lambda; T_V) = M_\kappa(\lambda), \quad \lambda > 0,$$

where

$$M_k := \binom{d + k - 1}{d - 1} + \binom{d + k - 2}{d - 1}, \quad k \in \mathbb{Z}^+,$$

with

$$\binom{m}{n} = \begin{cases} \frac{m!}{(m-n)!n!} & \text{if } m \geq n, \\ 0 & \text{if } m < n, \end{cases}$$

and

$$\kappa(\lambda) := \# \left\{ k \in \mathbb{Z}^+_+ | bc^{2k+d} > \lambda \right\}, \quad \lambda > 0.$$
Thank you!