Eigenvalue Estimates for Quantum Graphs

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The Laplacian on metric graphs

- Consider a metric graph $\Gamma = (\mathcal{E}(\Gamma), \mathcal{V}(\Gamma))$, $\mathcal{V}(\Gamma) = \{v_i\}_{i \in I}$, $\mathcal{E}(\Gamma) = \{e_j\}_{j \in J}$, where each edge is identified with an interval, $e_j \sim (a_j, b_j)$
- We allow multiple parallel edges between vertices and loops, but our edges will be finite
- Take the Laplacian with “natural” boundary conditions on $\Gamma$: models heat diffusion on a graph: Laplacian (i.e. second derivative) on each edge-interval; continuity plus Kirchhoff condition at the vertices: flow in equals flow out, i.e. the sum of the normal derivatives is zero
- The vertex conditions are generally encoded in the domain of the operator / associated form
The Laplacian on metric graphs

Formally

\[ H^1(\Gamma) := \{ u : \Gamma \to \mathbb{R} : u|_{e_j} \in H^1(e_j) \sim H^1(a_j, b_j) \text{ for all edges } e_j \]

and if \( e_1 \sim (a_1, b_1) \) and \( e_2 \sim (a_2, b_2) \) share a common vertex \( b_1 \sim a_2 \), then \( u(b_1) = u(a_2) \} \hookrightarrow C(\Gamma) \)

Define a bilinear form \( a : H^1(\Gamma) \to \mathbb{R} \) by

\[
a(u, v) := \int_{\Gamma} \nabla u \cdot \nabla v = \sum_j \int_{e_j} u'|_{e_j} v'|_{e_j}, \quad u, v \in H^1(\Gamma)
\]

Call the associated operator in \( L^2(\Gamma) \) the Laplacian with natural boundary conditions or “Kirchhoff Laplacian” , \(-\Delta_{\Gamma}\)
The eigenvalues of the Laplacian

Assume $\Gamma$ is connected and consists of finitely many edges and vertices, and each edge has finite length. Then $-\Delta_{\Gamma}$ has a sequence of eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \rightarrow \infty$$

- $\lambda_0 = 0$ with constant functions as eigenfunctions
- Resembles the Neumann Laplacian
  - If $\Gamma$ consists of a single edge connecting two vertices, it is the Neumann Laplacian on an interval
  - If $\Gamma$ consists of a single edge connecting the one vertex (i.e. a loop), it is the Laplace-Beltrami operator on a flat circle

Question (“Spectral geometry”)
How do the eigenvalues depend on (properties of) $\Gamma$?
Spectral geometry on domains/manifolds

- Background: “shape optimisation” on domains or manifolds: which domain optimises an eigenvalue (or combination) among all domains with a given property?

- Classical example: the Theorem of (Rayleigh–) Faber–Krahn: for the Dirichlet Laplacian

\[-\Delta u = \lambda u \quad \text{in } \Omega \subset \mathbb{R}^d,
\]
\[u = 0 \quad \text{on } \partial \Omega,
\]

with eigenvalues \(0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \ldots\),

**Theorem**

Let \(B \subset \mathbb{R}^d\) be a ball with the same volume as \(\Omega\). Then \(\lambda_1(B) \leq \lambda_1(\Omega)\) with equality iff \(\Omega\) is (essentially) a ball.

Why? Classical *isoperimetric inequality* plus variational characterisation of \(\lambda_1\) plus geometry and analysis.
We will concentrate (mostly) on $\lambda_1$, i.e. the spectral gap.

Variational characterisation:

$$\lambda_1(\Gamma) = \inf \left\{ \frac{\|\nabla u\|^2_{L^2(\Gamma)}}{\|u\|^2_{L^2(\Gamma)}} : 0 \neq u \in H^1(\Gamma), \int_{\Gamma} u = 0 \right\}$$

“Volume” is the total length $L(\Gamma) := \sum_j |e_j| = \sum_j (b_j - a_j)$.

Rescaling $\Gamma$ rescales the eigenvalues accordingly.

Theorem (Faber–Krahn-type inequality for graphs; S. Nicaise, 1986; L. Friedlander, 2005; P. Kurasov & S. Naboko, 2013)

$$\lambda_1(\Gamma) \geq \frac{\pi^2}{L^2} = \lambda_1(\text{line of length } L).$$

Equality holds iff $\Gamma$ is a line.

In fact $\lambda_k(\Gamma) \geq \frac{\pi^2(k+1)^2}{4L^2}, \ k \geq 1$ (Friedlander).
What properties of $\Gamma$ should $\lambda_1(\Gamma)$ depend on?

- Length $L(\Gamma)$
- “Surface area of the boundary”: Number of vertices $V(\Gamma)$
- Also number of edges $E(\Gamma)$?
- Diameter: $D(\Gamma) = \sup_{x,y \in \Gamma} \text{dist}(x,y)$
  Distance is measured along paths within $\Gamma$
- The edge connectivity $\eta$
- The Betti number $\beta = E - V + 1$
- The Cheeger constant of $\Gamma$

...  

How? Basic variational techniques become much more powerful in one dimension!
“Surgery” on graphs

Recall the variational characterisation

\[ \lambda_1(\Gamma) = \inf \left\{ \frac{\|\nabla u\|_{L^2(\Gamma)}^2}{\|u\|_{L^2(\Gamma)}^2} : 0 \neq u \in H^1(\Gamma), \int_\Gamma u = 0 \right\}, \]

where

\[ H^1(\Gamma) = \{ u : \Gamma \to \mathbb{R} : u|_{e_j} \in H^1(e_j) \sim H^1(a_j, b_j) \text{ for all edges } e_j \]
and if \( e_1 \sim (a_1, b_1) \) and \( e_2 \sim (a_2, b_2) \) share a common vertex \( b_1 \sim a_2 \), then \( u(b_1) = u(a_2) \} \).

- Attaching a pendant edge (or graph) to a vertex lowers \( \lambda_1 \) ("monotonicity" with respect to graph inclusion)
- Lengthening a given edge lowers \( \lambda_1 \) (essentially the same)
- Creating a new graph by identifying two vertices raises \( \lambda_1 \)
- Adding a new edge between two vertices is a “global” change; the eigenvalue can increase or decrease

Similar principles even hold for the higher eigenvalues \( \lambda_k \).
Theorem (K.-Kurasov-Malenová-Mugnolo, 2015)

Denote by $E$ the number of edges of $\Gamma$. Then

$$\lambda_1(\Gamma) \leq \frac{\pi^2 E^2}{L^2}.$$  

Equality holds iff $\Gamma$ is equilateral and there is an eigenfunction equal to zero on all vertices of $\Gamma$.

Proof: elementary. Use the surgery principles to reduce to a class of maximisers ("flower graphs", $E$ loops connected to a single vertex) and analyse this class.

Interesting phenomenon: there are two “types” of maximisers: flower graphs and “pumpkin” (aka “mandarin”) graphs.

In fact $\lambda_k(\Gamma) \leq \frac{\pi^2 E^2(k+1)^2}{4L^2}$ if $\Gamma$ is a “tree” (Rohleder, 2016)
Bounds and non-bounds on $\lambda_1(\Gamma)$

- Fix $L$ and $V$ (number of vertices, instead of number of edges). Then $\lambda_1 \to \infty$ is possible.
- Fix $E$ and $V$. Then $\lambda_1 \to 0$ and $\lambda_1 \to \infty$ are possible. (Rescaling!)

The Cheeger constant

$$h(\Gamma) = \inf_{S \subset \Gamma_{\text{open}}} \frac{\# \partial S}{\min\{\vert S\vert, \vert S^c\vert\}}.$$  

**Theorem**

$$\frac{h(\Gamma)^2}{4} \leq \lambda_1(\Gamma) \leq \frac{\pi^2 E^2 h(\Gamma)^2}{4}.$$  

Optimality of the bounds??
What about diameter $D$?

Example (K.-Kurasov-Malenová-Mugnolo, 2015)

There exists a sequence of graphs $\Gamma_n$ ("flower dumbbells") with $D(\Gamma_n) = 1$, $V(\Gamma_n) = 2$ and $\lambda_1(\Gamma_n) \to 0$.

This can be established via a simple test function argument. Much harder (and less obvious) is

Example (K.-Kurasov-Malenová-Mugnolo, 2015)

There exists a sequence of graphs $\Gamma_n$ ("pumpkin chains") with $D(\Gamma_n) = 1$ and $\lambda_1(\Gamma_n) \to \infty$.

Remark

$\lambda_1(\Gamma_n) \to \infty$ is a "global" property of $\Gamma_n$: attach a fixed pendant edge $e$ of length $\ell > 0$ to each $\Gamma_n$ to form a new graph $\tilde{\Gamma}_n$, then $\lambda_1(\tilde{\Gamma}_n) \leq \frac{\pi^2}{\ell^2}$ for all $n$. (Surgery principle: attaching the pendant graph $\Gamma_n$ to $e$ can only lower the eigenvalue of $e$!)
More bounds on $\lambda_1(\Gamma)$?

**Theorem (K.-Kurasov-Malenová-Mugnolo, 2015)**

If $\Gamma$ has diameter $D$, $E$ edges and $V \geq 2$ vertices, then

$$\lambda_1(\Gamma) \leq \frac{\pi^2}{D^2} (V + 1)^2$$

and

$$\frac{\pi^2}{D^2 E^2} \leq \lambda_1(\Gamma) \leq \frac{4\pi^2 E^2}{D^2},$$

with equality in the lower bound if $\Gamma$ is a path and in the upper bound if $\Gamma$ is a loop.
More bounds on $\lambda_1(\Gamma)$?

Edge connectivity $\eta$ is the minimum number of “cuts” needed to make $\Gamma$ disconnected. Rules:
- Vertices cannot be cut;
- Each edge can only be cut once.

**Theorem (Band–Lévy ’16, Berkolaiko-K.-Kurasov-Mugnolo, ’16)**

Suppose $\eta(\Gamma) \geq 2$. Then

$$\lambda_1(\Gamma) \geq \frac{4\pi^2}{L^2}.$$  

(A refinement of Nicaise et al; the proof is a refinement of Friedlander’s rearrangement method.) A further refinement:

**Theorem (Berkolaiko-K.-Kurasov-Mugnolo, ’16)**

Suppose $\ell_{\text{max}}$ denotes the length of the longest edge of $\Gamma$. Then

$$\lambda_1(\Gamma) \geq \frac{\pi^2 \eta^2}{(L + \ell_{\text{max}}(\eta - 2)_+)^2}.$$
Thank you for your attention!