Branching form of the resolvent at threshold for discrete Laplacians

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Introduction: Discrete Laplacian

∘ Thresholds generated by critical values

For any function $u\colon \mathbb{Z}^d \to \mathbb{C}$ define $\triangle u\colon \mathbb{Z}^d \to \mathbb{C}$ by

$$(\triangle u)[n] = \sum_{j=1}^{d} \Big(u[n+e_j] + u[n-e_j] - 2u[n] \Big) \quad \text{for } n \in \mathbb{Z}^d.$$

The operator $H_0 = -\triangle$ is bounded and self-adjoint on $\mathcal{H} = \ell^2(\mathbb{Z}^d)$.
It has spectrum

$$\sigma(H_0) = \sigma_{ac}(H_0) = [0, 4d],$$

and thresholds

$$\tau(H_0) = \{0, 4, \dots, 4d\}.$$

Let $\widehat{\mathcal{H}} = L^2(\mathbb{T}^d)$, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, and define the Fourier transform $\mathcal{F}\colon \mathcal{H} \to \widehat{\mathcal{H}}$ and its inverse $\mathcal{F}^*\colon \widehat{\mathcal{H}} \to \mathcal{H}$ by

$$(\mathcal{F}u)(\theta) = (2\pi)^{-d/2} \sum_{n\in\mathbb{Z}^d} e^{-in\theta} u[n],$$

$$(\mathcal{F}^* f)[n] = (2\pi)^{-d/2} \int_{\mathbb{T}^d} e^{in\theta} f(\theta) d\theta,$$

and then

$$\mathcal{F} H_0 \mathcal{F}^* = \Theta(\theta) = 2d - 2\cos\theta_1 - \cdots - 2\cos\theta_d.$$

Since $\partial_j \Theta(\theta) = 2\sin\theta_j$, the critical points of signature (p, q) are

$$\gamma(p, q) = \left\{ \theta \in \{0, \pi\}^d;\ \#\{\theta_j = 0\} = p,\ \#\{\theta_j = \pi\} = q \right\}.$$

Hence the critical values $0, 4d$ are thresholds of elliptic type, and $4, \ldots, 4(d-1)$ are those of hyperbolic type.

∘ Purpose: Asymptotic expansion of resolvent

Q. Can we compute an asymptotic expansion of the resolvent

$$R_0(z) = (H_0 - z)^{-1} \sim ?? \quad \text{as } z \to 4q \text{ for } q = 0, 1, \ldots, d?$$

Note that the resolvent $R_0(z)$ has a convolution kernel:

$$R_0(z)u = k(z, \,\cdot\,) * u; \quad k(z, n) = (2\pi)^{-d} \int_{\mathbb{T}^d} \frac{e^{in\theta}}{\Theta(\theta) - z}\, d\theta.$$

A. Yes. By localizing around $\gamma(p, q) \subset \mathbb{T}^d$ and changing variables the situation reduces to that for an ultra-hyperbolic operator.

- As far as we know, an explicit asymptotics of $R_0(z)$ around a threshold seems to have been open except for 0 and $4d$.

Ultra-hyperbolic operator (a model operator)

Consider an ultra-hyperbolic operator on $\mathbb{R}^d$:

$$\Box = \partial_1^2 + \cdots + \partial_p^2 - \partial_{p+1}^2 - \cdots - \partial_{p+q}^2; \quad p, q \geq 0,\ d = p + q.$$

The operator $H_0 = -\Box$ is self-adjoint on $\mathcal{H} = L^2(\mathbb{R}^d)$ with

$$\mathcal{D}(H_0) = \{u \in \mathcal{H};\ \Box u \in \mathcal{H} \text{ in the distributional sense}\}.$$

It has spectrum

$$\sigma(H_0) = \sigma_{\mathrm{ac}}(H_0) = \begin{cases} [0, \infty) & \text{if } (p, q) = (d, 0), \\ (-\infty, 0] & \text{if } (p, q) = (0, d), \\ \mathbb{R} & \text{otherwise}, \end{cases}$$

and a single threshold

$$\tau(H_0) = \{0\}.$$

Using the Fourier transform $\mathcal{F}\colon \mathcal{H} \to \mathcal{H}$ and its inverse $\mathcal{F}^*\colon \mathcal{H} \to \mathcal{H}$, we can write

$$\mathcal{F} H_0 \mathcal{F}^* = \Xi(\xi) = \xi'^2 - \xi''^2; \qquad \xi = (\xi', \xi'') \in \mathbb{R}^p \oplus \mathbb{R}^q.$$

The only critical point is $\xi = 0$, and the associated critical value, or a threshold 0 is said to be

1. of elliptic type if $(p, q) = (d, 0)$ or $(0, d)$;

2. of hyperbolic type otherwise.

Q′. Can we compute an asymptotic expansion of the resolvent

$$R_0(z) = (H_0 - z)^{-1} \sim ?? \quad \text{as } z \to 0?$$

A′. Yes. In particular, square root, logarithm and dilogarithm branchings show up, depending on parity of (p, q).

∘ Square root, logarithm and dilogarithm

We always choose branches of $\sqrt{w}$ and $\log w$ such that

$$\operatorname{Im}\sqrt{w} > 0 \quad \text{for } w \in \mathbb{C} \setminus [0,\infty),$$
$$-\pi < \operatorname{Im}\log w < \pi \quad \text{for } w \in \mathbb{C} \setminus (-\infty,0],$$

respectively. In addition, let us set for $w \in \mathbb{C} \setminus [1,\infty)$

$$\mathrm{Li}_1(w) = -\log(1-w), \qquad \mathrm{Li}_2(w) = \int_0^w \frac{\mathrm{Li}_1(\lambda)}{\lambda}\, d\lambda,$$

which have the Taylor expansions: For $|w| < 1$

$$\mathrm{Li}_1(w) = \sum_{k=1}^{\infty} \frac{w^k}{k}, \qquad \mathrm{Li}_2(w) = \sum_{k=1}^{\infty} \frac{w^k}{k^2}.$$

∘ Elliptic operator in odd dimensional space

Theorem. *Let* d *be odd,* $(p, q) = (d, 0)$*, and* $\gamma > 0$*. Then*

$$k_\gamma(z, x) = \frac{i\pi}{2}(\sqrt{z})^{d-2} e(\sqrt{z}x) - \frac{1}{2}\int_{\widetilde{\Gamma}(\gamma)} \frac{\rho^{d-1} e(\rho x)}{\rho^2 - z}\, d\rho,$$

where $\widetilde{\Gamma}(r) = \{re^{i\theta} \in \mathbb{C};\ \theta \in [0, \pi]\}$.

∘ Elliptic operator in even dimensional space

Theorem. *Let* d *be even,* $(p, q) = (d, 0)$*, and* $\gamma > 0$*. Then*

$$k_\gamma(z, x) = -\frac{1}{2}(\sqrt{z})^{d-2} e(\sqrt{z}x)\,\mathrm{Li}_1\left(\frac{\gamma^2}{z}\right) + \frac{1}{2}\int_0^{\gamma^2} \frac{(\sqrt{\lambda})^{d-2} e(\sqrt{\lambda}x) - (\sqrt{z})^{d-2} e(\sqrt{z}x)}{\lambda - z}\, d\lambda.$$

Proposition. *1. The function* $e(\zeta)$ *is even and entire in* $\zeta \in \mathbb{C}^d$*, and*

$$e(\zeta) = \sum_{\alpha \in \mathbb{Z}_+^d} e_\alpha \zeta^{2\alpha}; \quad e_\alpha = \frac{2}{2^d \pi^{d/2}} \frac{(-1/4)^{|\alpha|}}{\alpha! \Gamma(|\alpha| + d/2)}.$$

2. For any $z \in \mathbb{C}$ *the function* $e(\sqrt{z}x)$ *satisfies the eigenequation*

$$(-\triangle - z) e(\sqrt{z}x) = 0; \quad \triangle = \square_{d,0}.$$

Here a branch of $\sqrt{z}$ *does not matter, since* $e(\zeta)$ *is even.*

◦ **Hyperbolic case with odd-even or even-odd signature**

Theorem. *Let* (p, q) *be odd-even or even-odd, and* $\gamma > 0$. *Then*

$$k_\gamma(z, x) = \frac{i\pi}{2}(\sqrt{z})^{d-2}\psi_+(\sqrt{z}x) + \chi_\gamma(z, x),$$

where

$$\chi_\gamma(z, x) = -\frac{1}{2}\int_{\Gamma(\gamma)} \frac{\tau^{d-1}\psi_+(\tau x)}{\tau^2 - z}\, d\tau + \frac{1}{2}\int_{\Gamma(\gamma)} \frac{\tau^{d-1}\psi_-(\tau x)}{\tau^2 + z}\, d\tau + \frac{1}{4}\int_{i^2\widetilde{\Gamma}(\gamma^2)} \frac{h_{+,\gamma}(\lambda, x)}{\lambda - z}\, d\lambda$$

with

$$h_{\pm,\gamma}(\tau^2, x) = \tau^{d-2}\int_{-\gamma}^{\gamma} \frac{f_\pm(\sigma/\tau, \tau x)}{\sigma}\, d\sigma.$$

∘ Hyperbolic case with even-even signature

Theorem. *Let* (p, q) *be even-even, and* $\gamma > 0$*. Then*

$$k_\gamma(z, x) = -\frac{1}{2}(\sqrt{z})^{d-2}\psi_+(\sqrt{z}x)\,\mathrm{Li}_1\left(\frac{\gamma^4}{z^2}\right) + \chi_\gamma(z, x),$$

where

$$\chi_\gamma(z, x) = \frac{1}{2}\left(\int_0^{\gamma^2} - \int_{-\gamma^2}^{0}\right)\frac{(\sqrt{\tau})^{d-2}\psi_+(\sqrt{\tau}x) - (\sqrt{z})^{d-2}\psi_+(\sqrt{z}x)}{\tau - z}\,d\tau$$
$$+ \frac{1}{2}\int_{-\gamma^2}^{\gamma^2}\frac{h_{+,\gamma}(\lambda, x)}{\lambda - z}\,d\lambda$$

with

$$h_{\pm,\gamma}(\tau^2, x) = \tau^{d-2}\int_\tau^\gamma \frac{f_\pm(\sigma/\tau, \tau x)}{\sigma}\,d\sigma - \tau^{d-2}\psi_\pm(\tau x).$$

◦ **Hyperbolic case with odd-odd signature**

Theorem. *Let* (p, q) *be odd-odd, and* $\gamma > 0$. *Then*

$$k_\gamma(z,x) = -\frac{1}{4}(\sqrt{z})^{d-2}\phi_+(\sqrt{z}x)\left[\mathrm{Li}_2\left(\frac{\gamma^2}{z}\right) - \mathrm{Li}_2\left(-\frac{\gamma^2}{z}\right)\right] + \chi_\gamma(z,x),$$

where

$$\begin{aligned}\chi_\gamma(z,x) = &\frac{1}{4}\int_0^{\gamma^2} \frac{(\sqrt{\lambda})^{d-2}\phi_+(\sqrt{\lambda}x) - (\sqrt{z})^{d-2}\phi_+(\sqrt{z}x)}{\lambda - z}\log\left(\frac{\gamma^2}{\lambda}\right)d\lambda \\ &+ \frac{1}{4}\int_{-\gamma^2}^{0} \frac{(\sqrt{\lambda})^{d-2}\phi_+(\sqrt{\lambda}x) - (\sqrt{z})^{d-2}\phi_+(\sqrt{z}x)}{\lambda - z}\log\left(-\frac{\gamma^2}{\lambda}\right)d\lambda \\ &+ \frac{1}{2}\int_{-\gamma^2}^{\gamma^2} \frac{h_{+,\gamma}(\lambda,x)}{\lambda - z}\,d\lambda\end{aligned}$$

with $h_{\pm,\gamma}(\tau^2, x) = \tau^{d-2}\int_\tau^\gamma \frac{f_\pm(\sigma/\tau, \tau x) - \phi_\pm(\tau x)}{\sigma}\,d\sigma.$

∘ Properties of $\phi_\pm(\zeta)$ and $\psi_\pm(\zeta)$

The functions $\phi_\pm(\zeta)$ and $\psi_\pm(\zeta)$ are entire in $\zeta \in \mathbb{C}^d$, and

$$\phi_\pm(\zeta) = \sum_{\alpha \in \mathbb{Z}_+^d} \zeta^{2\alpha} \left(\sum_{a \in J_\alpha} f_{\pm,\alpha,a} \right),$$

$$\psi_\pm(\zeta) = \sum_{\alpha \in \mathbb{Z}_+^d} \zeta^{2\alpha} \left(\sum_{a \in I_\alpha \setminus J_\alpha} \frac{\left[i^{2|\alpha|-2|a|+d-2} - 1 \right] f_{\pm,\alpha,a}}{2|\alpha| - 2|a| + d - 2} \right),$$

where

$$I_\alpha = \left\{ a \in \mathbb{Z}_+^2;\ 0 \le a \le \left(2|\alpha'| + p - 1, 2|\alpha''| + q - 1 \right) \right\},$$

$$J_\alpha = \left\{ a \in I_\alpha;\ |a| = |\alpha| + (d-2)/2 \right\},$$

$$f_{\pm,\alpha,a} = \frac{(\pm 1)^{a'} (\mp 1)^{a''} e'_{\alpha'} e''_{\alpha''}}{2^{2|\alpha|+d-2}} \binom{2|\alpha'| + p - 1}{a'} \binom{2|\alpha''| + q - 1}{a''}.$$

◦ Properties of $\phi_\pm(\zeta)$ and $\psi_\pm(\zeta)$, continued

The functions $\phi_\pm(\zeta)$ and $\psi_\pm(\zeta)$ satisfy

1. $\phi_\pm(\zeta) = 0$ and $\psi_\pm(\zeta) = \psi_\pm(-\zeta)$ if (p, q) is odd-even or even-odd;

2. $\phi_\pm(\zeta) = 0$ and $\psi_\pm(\zeta) = -i^{d-2}\psi_\mp(i\zeta)$ if (p, q) is even-even;

3. $\phi_\pm(\zeta) = i^{d-2}\phi_\mp(i\zeta)$ and $\psi_\pm(\zeta) = 0$ if (p, q) is odd-odd.

In addition, for any $z \in \mathbb{C}$

$$(-\Box \mp z)\phi_\pm(\sqrt{z}x) = 0, \qquad (-\Box \mp z)\psi_\pm(\sqrt{z}x) = 0.$$

∘ Outline of the results for ultra-hyperbolic operator

Theorem. *1. If* (p, q) *is odd-even or even-odd, there exist operators* $F(z), G(z)$ *analytic at* $z = 0$ *such that*

$$R_0(z) = F(z)\sqrt{z} + G(z).$$

2. If (p, q) *is even-even, there exist operators* $F(z), G(z)$ *analytic at* $z = 0$ *such that*

$$R_0(z) = F(z)\mathrm{Li}_1\left(\frac{1}{z}\right) + G(z).$$

3. If (p, q) *is odd-odd, there exist operators* $F(z), G(z)$ *analytic at* $z = 0$ *such that*

$$R_0(z) = F(z)\left[\mathrm{Li}_2\left(\frac{1}{z}\right) - \mathrm{Li}_2\left(-\frac{1}{z}\right)\right] + G(z).$$

◦ "Very rough" strategy for proof

The resolvent has a limiting convolution expression

$$(R_0(z)u)(x) = \lim_{\gamma\to\infty}\int_{\mathbb{R}^d} k_\gamma(z, x-y)u(y)\,dy \quad \text{for } u \in \mathcal{S}(\mathbb{R}^d);$$

$$k_\gamma(z,x) = (2\pi)^{-d}\int_{|\xi'|+|\xi''|<\gamma} \frac{e^{ix\xi}}{\xi'^2 - \xi''^2 - z}\,d\xi.$$

It suffices to expand the kernel $k_\gamma(z,x)$, since it contains all the singular part of $R_0(z)$.

If we move on to the spherical or hyperbolic coordinates, a singular part of $k_\gamma(z,x)$ takes, more or less, the *standard form*

$$I = \int_0^\gamma \frac{a(\rho)}{\rho^2 - z}\,d\rho.$$

There could appear only the following three types of $a(\rho)$:

- If $a(\rho) = 2b(\rho^2)$ with b analytic, then

$$I = \int_{-\gamma}^{\gamma} \frac{b(\rho^2)}{(\rho - \sqrt{z})(\rho + \sqrt{z})} d\rho = i\pi \frac{b(z)}{\sqrt{z}} - \int_{|z|=\gamma,\, \mathrm{Im}\, z \geq 0} \frac{b(\rho^2)}{\rho^2 - z} d\rho.$$

- If $a(\rho) = 2\rho b(\rho^2)$ with b analytic, then

$$I = \int_0^{\gamma} \frac{b(\lambda)}{\lambda - z} d\lambda = \int_0^{\gamma} \frac{b(z)}{\lambda - z} d\lambda + \int_0^{\gamma} \frac{b(\lambda) - b(z)}{\lambda - z} d\lambda$$

- If $a(\rho) = 2\rho b(\rho^2)(\log \rho^2)$ with b analytic, then

$$I = \int_0^{\gamma} \frac{b(\lambda) \log \lambda}{\lambda - z} d\lambda = \int_0^{\gamma} \frac{b(z) \log \lambda}{\lambda - z} d\lambda + \int_0^{\gamma} \frac{[b(\lambda) - b(z)] \log \lambda}{\lambda - z} d\lambda$$

Precise results for discrete Laplacian

◦ Localization around critical points

Note that $R_0(z)$ has a convolution kernel:

$$k(z,n) = (2\pi)^{-d} \int_{\mathbb{T}^d} \frac{e^{in\theta}}{\Theta(\theta) - z}\, d\theta.$$

Denote the set of all the critical points of signature (p,q) by

$$\gamma(p,q) = \{\theta^{(l)}\}_{l=1,\ldots,L}; \quad L = \#\gamma(p,q) = \binom{d}{p} = \binom{d}{q},$$

Take neighborhoods $U_l \subset \mathbb{T}^d$ of $\theta^{(l)}$, and then decompose

$$k(z,n) = k_0(z,n) + k_1(z,n) + \cdots + k_L(z,n);$$

$$k_l(z,n) = (2\pi)^{-d} \int_{U_l} \frac{e^{in\theta}}{\Theta(\theta) - z}\, d\theta \quad \text{for } l = 1, \ldots, L.$$

We let, e.g.,

$$\theta^{(1)} = (0, \ldots, 0, \pi, \ldots, \pi) \in \mathbb{T}^p \times \mathbb{T}^q = \mathbb{T}^d.$$

Take local coordinates $\xi(\theta) = \xi^{(1)}(\theta)$ around $\theta^{(1)} \in \mathbb{T}^d$:

$$\xi_j(\theta) = \begin{cases} 2\sin(\theta_j/2) & \text{for } j = 1, \ldots, p, \\ 2\cos(\theta_j/2) & \text{for } j = p+1, \ldots, p+q, \end{cases}$$

and set

$$U_1 = \{\theta \in \mathbb{T}^d;\ |\xi'(\theta)| + |\xi''(\theta)| < 2\}.$$

Then we can write

$$k_1(z,n) = (2\pi)^{-d} \int_{|\xi'|+|\xi''|<2} \frac{e^{in\theta(\xi)}}{\xi'^2 - \xi''^2 - (z-4q)} \frac{d\xi}{\prod_{j=1}^d (1-\xi_j^2/4)^{1/2}}.$$

We will do the same construction for $k_2(z,n), \ldots, k_L(z,n)$.

Theorem. *Let* $(p, q) = (d, 0)$.

1. *If* d *is odd, there exists a function* $\chi(z, n)$ *analytic in* $z \in \Delta(4)$ *such that*

$$k(z, n) = \frac{i\pi}{2}(\sqrt{z})^{d-2} e(\sqrt{z}, n) + \chi(z, n).$$

2. *If* d *is even, there exists a function* $\chi(z, n)$ *analytic in* $z \in \Delta(4)$ *such that*

$$k(z, n) = -\frac{1}{2}(\sqrt{z})^{d-2} e(\sqrt{z}, n) \operatorname{Li}_1\left(\frac{1}{z}\right) + \chi(z, n).$$

Proposition. *1. The function* $e(\rho, n)$ *is analytic in* $\rho \in \Delta(2)$*, and*

$$e(\rho, n) = \frac{2}{2^d \pi^{d/2}} \sum_{k=0}^{\infty} \rho^{2k} \sum_{\alpha \in \mathbb{Z}_+^d, |\alpha|=k} \frac{\prod_{j=1}^{d} (1/2 - n_j)_{\alpha_j} (1/2 + n_j)_{\alpha_j}}{4^{|\alpha|} \alpha! \Gamma(|\alpha| + d/2)},$$

where $(\nu)_k := \Gamma(\nu + k)/\Gamma(\nu)$ *is the Pochhammer symbol. In particular,* $e(z, n)$ *can be expressed by the Lauricella function:*

$$e(\rho, n) = \frac{2}{2^d \pi^{d/2} \Gamma(d/2)} F_B^{(d)} \Big(\frac{1}{2} - n_1, \dots, \frac{1}{2} - n_d, \frac{1}{2} + n_1, \dots, \frac{1}{2} + n_d; \frac{d}{2}; \frac{\rho^2}{4}, \dots, \frac{\rho^2}{4} \Big).$$

2. For any $z \in \Delta(4)$*, as a function in* $n \in \mathbb{Z}^d$*,*

$$(-\triangle - z) e(\sqrt{z}, n) = 0.$$

Theorem. *1. If* (p, q) *is odd-even or even-odd, then there exists* $\chi(\cdot, n) \in C^\omega(\Delta(4))$ *such that*

$$k(w + 4q, n) = \frac{i\pi}{2}(\sqrt{w})^{d-2} \sum_{l=1}^{L} \psi_+^{(l)}(\sqrt{w}, n) + \chi(w, n).$$

2. If (p, q) *is even-even, there exists* $\chi(\cdot, n) \in C^\omega(\Delta(4))$ *such that*

$$k(w + 4q, n) = -\frac{1}{2}(\sqrt{w})^{d-2} \operatorname{Li}_1\left(\frac{16}{w^2}\right) \sum_{l=1}^{L} \psi_+^{(l)}(\sqrt{w}, n) + \chi(w, n).$$

3. If (p, q) *is odd-odd, there exists* $\chi(\cdot, n) \in C^\omega(\Delta(4))$ *such that*

$$k(w + 4q, n) = -\frac{1}{4}(\sqrt{w})^{d-2} \left[\operatorname{Li}_2\left(\frac{4}{w}\right) - \operatorname{Li}_2\left(-\frac{4}{w}\right)\right] \sum_{l=1}^{L} \phi_+^{(l)}(\sqrt{w}, n) + \chi(w, n).$$

◦ Properties of $\phi_\pm(\tau,n)$ and $\psi_\pm(\tau,n)$

The functions $\phi_\pm(\tau,n)$ and $\psi_\pm(\tau,n)$ are analytic in $\tau \in \Delta(2)$, and

$$\phi_\pm(\tau,n) = \sum_{\alpha\in\mathbb{Z}_+^2} \tau^{2|\alpha|}\Big(\sum_{a\in J_\alpha} f_{\pm,\alpha,a}[n]\Big),$$

$$\psi_\pm(\tau,n) = \sum_{\alpha\in\mathbb{Z}_+^2} \tau^{2|\alpha|}\Big(\sum_{a\in I_\alpha\setminus J_\alpha} \frac{\left[i^{2|\alpha|-2|a|+d-2}-1\right] f_{\pm,\alpha,a}[n]}{2|\alpha|-2|a|+d-2}\Big),$$

where

$$I_\alpha = \Big\{a \in \mathbb{Z}_+^2;\ 0 \le a \le \big(2\alpha'+p-1, 2\alpha''+q-1\big)\Big\},$$

$$J_\alpha = \Big\{a \in I_\alpha;\ |a| = |\alpha| + (d-2)/2\Big\}.$$

$$f_{\pm,\alpha,a}[n] = \frac{(\pm 1)^{a'}(\mp 1)^{a''}}{2^{2|\alpha|+d-2}}\binom{2\alpha'+p-1}{a'}\binom{2\alpha''+q-1}{a''} e'_{\alpha'}[n'] e''_{\alpha''}[n''].$$

◦ Properties of $\phi_\pm(\tau, n)$ and $\psi_\pm(\tau, n)$, continued

The functions $\phi_\pm(\tau, n)$ and $\psi_\pm(\tau, n)$ satisfy

1. $\phi_\pm(\tau, n) = 0$, $\psi_\pm(\tau, n) = \psi_\pm(-\tau, n)$ and $\psi_\pm(\tau, n) = \psi_\pm(\tau, -n)$ if (p, q) is odd-even or even-odd;

2. $\phi_\pm(\tau, n) = 0$, $\psi_\pm(\tau, n) = -i^{d-2}\psi_\mp(i\tau, n)$ and $\psi_\pm(\tau, n) = \psi_\pm(\tau, -n)$ if (p, q) is even-even;

3. $\phi_\pm(\tau, n) = i^{d-2}\phi_\mp(i\tau, n)$, $\phi_\pm(\tau, n) = \phi_\pm(\tau, -n)$ and $\psi_\pm(\tau, n) = 0$ if (p, q) is odd-odd.

In addition, for any $w \in \Delta(4)$, as functions in $n \in \mathbb{Z}^d$,

$$(-\triangle \mp w)\phi_\pm(\sqrt{w}, n) = 0, \qquad (-\triangle \mp w)\psi_\pm(\sqrt{w}, n) = 0.$$