n-particle quantum statistics on graphs

Jon Harrison[1], J.P. Keating[2], J.M. Robbins[2] and A. Sawicki[2]

[1]Baylor University, [2]University of Bristol

QMath13 – 10/16

Outline

1. Quantum statistics
2. Statistics on graphs
3. 3-connected graphs

Quantum statistics

Single particle space configuration space X.

Two particle statistics - alternative approaches:

- Quantize $X^{\times 2}$ and restrict Hilbert space to the symmetric or anti-symmetric subspace.

$$\psi(x_1, x_2) = \pm\psi(x_2, x_1) \tag{1}$$

Bose-Einstein/Fermi-Dirac statistics.

Quantum statistics

Single particle space configuration space X.

Two particle statistics - alternative approaches:

- Quantize $X^{\times 2}$ and restrict Hilbert space to the symmetric or anti-symmetric subspace.

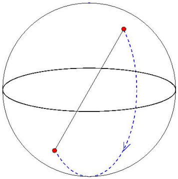
$$\psi(x_1, x_2) = \pm\psi(x_2, x_1) \tag{1}$$

Bose-Einstein/Fermi-Dirac statistics.

- (Leinaas and Myrheim '77)
Treat particles as indistinguishable, $\psi(x_1, x_2) \equiv \psi(x_2, x_1)$.
Quantize two particle configuration space.

Bose-Einstein and Fermi-Dirac statistics

Two indistinguishable particles in $\mathbb{R}^3$. At constant separation relative coordinate lies on projective plane.



Exchanging particles corresponds to rotating relative coordinate around closed loop p.

p is not contractible but p^2 is contractible.

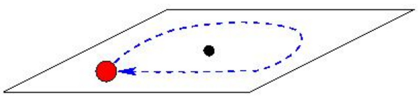
To associate a phase factor $\mathrm{e}^{\mathrm{i}\theta}$ to p requires $(\mathrm{e}^{\mathrm{i}\theta})^2 = 1$.

Quantizing configuration space with phase π corresponds to *Fermi-Dirac statistics* and phase 0 to *Bose-Einstein statistics*.

Anyon statistics

Pair of indistinguishable particles in $\mathbb{R}^2$.

- Particles not coincident.
- Relative position coordinate in $\mathbb{R}^2 \setminus \mathbf{0}$.
- Exchange paths are closed loops about $\mathbf{0}$ in relative coordinate.
- *Any* phase factor $e^{i\theta}$ can be associated with a primitive exchange path.



Definition

Configuration space of n indistinguishable particles in X,

$$C_n(X) = (X^{\times n} - \Delta_n)/S_n$$

where $\Delta_n = \{x_1, \ldots, x_n | x_i = x_j \text{ for some } i \neq j\}$.

Definition

Configuration space of n indistinguishable particles in X,

$$C_n(X) = (X^{\times n} - \Delta_n)/S_n$$

where $\Delta_n = \{x_1, \ldots, x_n | x_i = x_j \text{ for some } i \neq j\}$.

1st homology groups of $C_n(\mathbb{R}^d)$:

- $H_1(C_n(\mathbb{R}^d)) = \mathbb{Z}_2$ for $d \geq 3$.
  2 abelian irreps. corresponding to Bose-Einstein & Fermi-Dirac statistics.

Definition

Configuration space of n indistinguishable particles in X,

$$C_n(X) = (X^{\times n} - \Delta_n)/S_n$$

where $\Delta_n = \{x_1, \dots, x_n | x_i = x_j \text{ for some } i \neq j\}$.

1st homology groups of $C_n(\mathbb{R}^d)$:

- $H_1(C_n(\mathbb{R}^d)) = \mathbb{Z}_2$ for $d \geq 3$.
  2 abelian irreps. corresponding to Bose-Einstein & Fermi-Dirac statistics.
- $H_1(C_n(\mathbb{R}^2)) = \mathbb{Z}$
  Any single phase θ can be associated to primitive exchange paths – anyon statistics.

Definition

Configuration space of n indistinguishable particles in X,

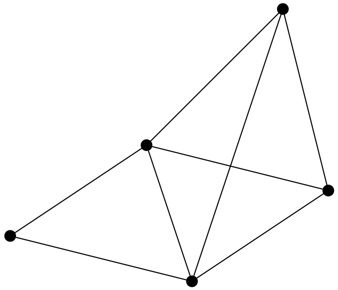
$$C_n(X) = (X^{\times n} - \Delta_n)/S_n$$

where $\Delta_n = \{x_1, \ldots, x_n | x_i = x_j \text{ for some } i \neq j\}$.

1st homology groups of $C_n(\mathbb{R}^d)$:

- $H_1(C_n(\mathbb{R}^d)) = \mathbb{Z}_2$ for $d \geq 3$.
  2 abelian irreps. corresponding to Bose-Einstein & Fermi-Dirac statistics.
- $H_1(C_n(\mathbb{R}^2)) = \mathbb{Z}$
  Any single phase θ can be associated to primitive exchange paths – anyon statistics.
- $H_1(C_n(\mathbb{R})) = 1$
  particles cannot be exchanged.

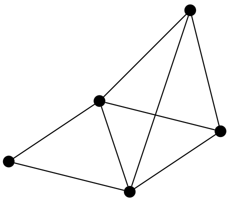
What happens on a graph where the underlying space has arbitrarily complex topology?



Graph connectivity

- Given a connected graph Γ a *k-cut* is a set of k vertices whose removal makes Γ disconnected.
- Γ is *k-connected* if the minimal cut is size k.
- **Theorem** (Menger) For a k-connected graph there exist at least k independent paths between every pair of vertices.

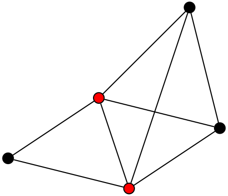
Example:



Graph connectivity

- Given a connected graph Γ a *k-cut* is a set of k vertices whose removal makes Γ disconnected.
- Γ is *k-connected* if the minimal cut is size k.
- **Theorem** (Menger) For a k-connected graph there exist at least k independent paths between every pair of vertices.

Example:

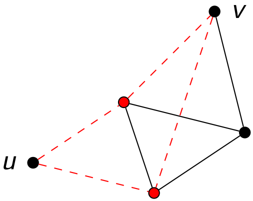


Two cut

Graph connectivity

- Given a connected graph Γ a *k-cut* is a set of k vertices whose removal makes Γ disconnected.
- Γ is *k-connected* if the minimal cut is size k.
- **Theorem** (Menger) For a k-connected graph there exist at least k independent paths between every pair of vertices.

Example:



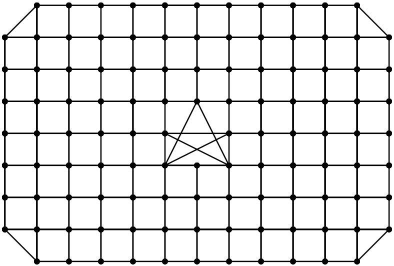
Two independent paths joining u and v.

Features of graph statistics

3-connected graphs: statistics only depend on whether the graph is planar (Anyons) or non-planar (Bosons/Fermions).

Features of graph statistics

3-connected graphs: statistics only depend on whether the graph is planar (Anyons) or non-planar (Bosons/Fermions).



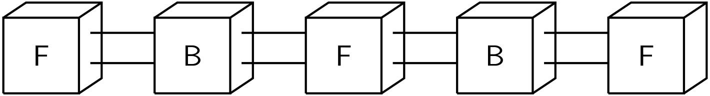
A planar lattice with a small section that is non-planar is locally planar but has Bose/Fermi statistics.

Features of graph statistics

2-connected graphs: statistics complex but independent of the number of particles.

Features of graph statistics

2-connected graphs: statistics complex but independent of the number of particles.



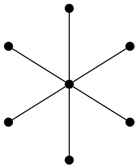
For example, one could construct a chain of 3-connected non-planar components where particles behave with alternating Bose/Fermi statistics.

Features of graph statistics

1-connected graphs: statistics depend on no. of particles n.

Features of graph statistics

1-connected graphs: statistics depend on no. of particles n.
Example, star with E edges.

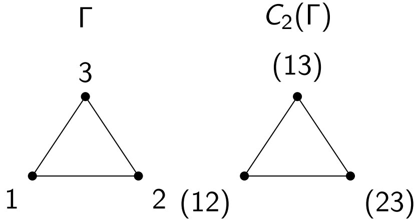


no. of anyon phases

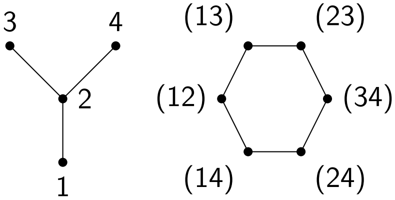
$$\binom{n+E-2}{E-1}(E-2)-\binom{n+E-2}{E-2}+1\ .$$

Basic cases

For 2 particles.



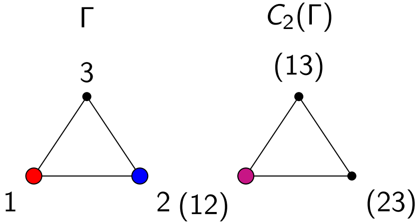
Exchange of 2 particles around loop c; one free phase ϕ_{c2}.



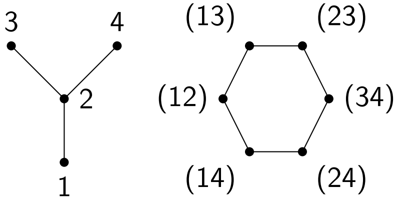
Exchange of 2 particles at Y-junction; one free phase ϕ_Y.

Basic cases

For 2 particles.



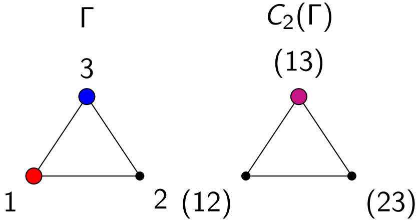
Exchange of 2 particles around loop c; one free phase ϕ_{c2}.



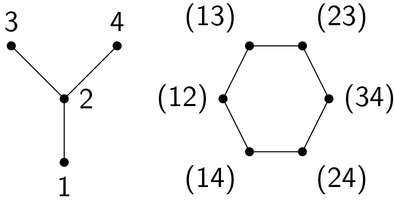
Exchange of 2 particles at Y-junction; one free phase ϕ_Y.

Basic cases

For 2 particles.



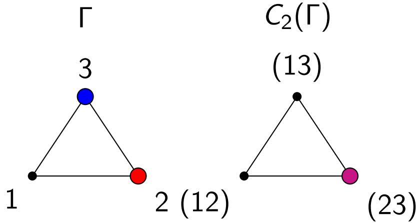
Exchange of 2 particles around loop c; one free phase ϕ_{c2}.



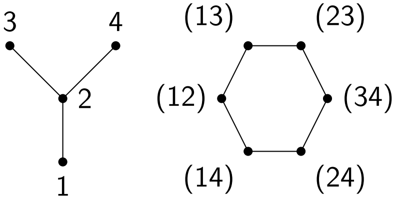
Exchange of 2 particles at Y-junction; one free phase ϕ_Y.

Basic cases

For 2 particles.



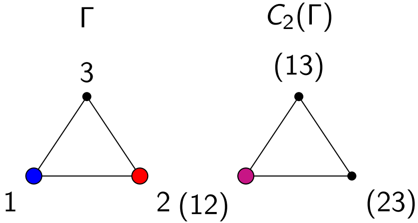
Exchange of 2 particles around loop c; one free phase ϕ_{c2}.



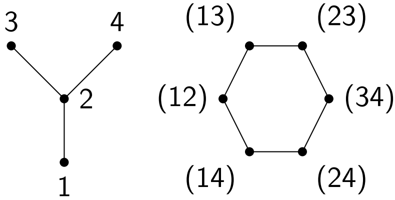
Exchange of 2 particles at Y-junction; one free phase ϕ_Y.

Basic cases

For 2 particles.



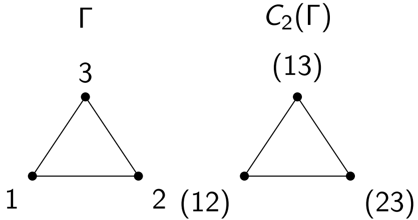
Exchange of 2 particles around loop c; one free phase ϕ_{c2}.



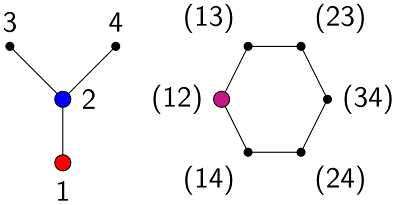
Exchange of 2 particles at Y-junction; one free phase ϕ_Y.

Basic cases

For 2 particles.



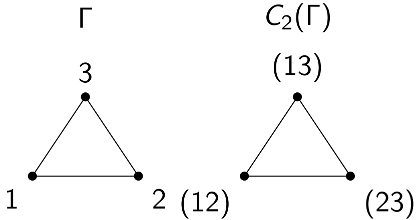
Exchange of 2 particles around loop c; one free phase ϕ_{c2}.



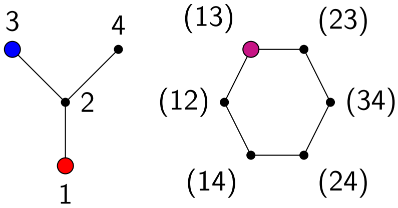
Exchange of 2 particles at Y-junction; one free phase ϕ_Y.

Basic cases

For 2 particles.



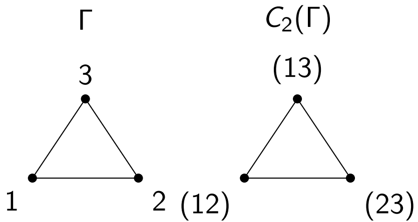
Exchange of 2 particles around loop c; one free phase ϕ_{c2}.



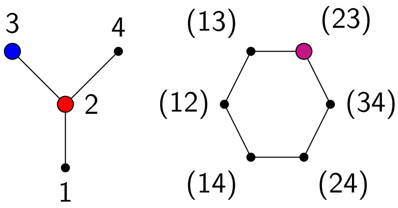
Exchange of 2 particles at Y-junction; one free phase ϕ_Y.

Basic cases

For 2 particles.



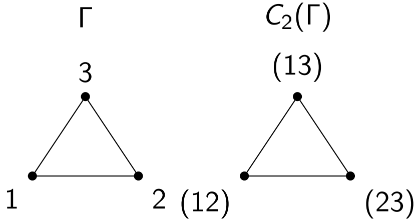
Exchange of 2 particles around loop c; one free phase ϕ_{c2}.



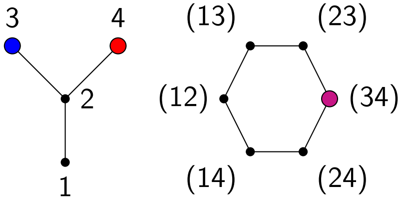
Exchange of 2 particles at Y-junction; one free phase ϕ_Y.

Basic cases

For 2 particles.



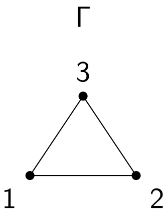
Exchange of 2 particles around loop c; one free phase ϕ_{c2}.

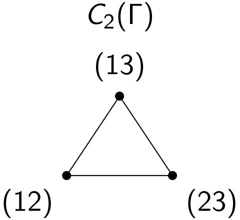


Exchange of 2 particles at Y-junction; one free phase ϕ_Y.

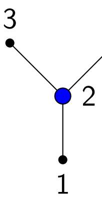
Basic cases

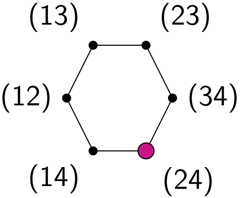
For 2 particles.





Exchange of 2 particles around loop c; one free phase ϕ_{c2}.

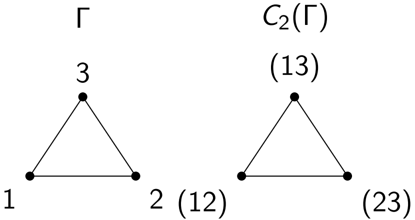




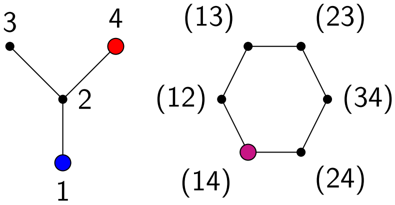
Exchange of 2 particles at Y-junction; one free phase ϕ_Y.

Basic cases

For 2 particles.



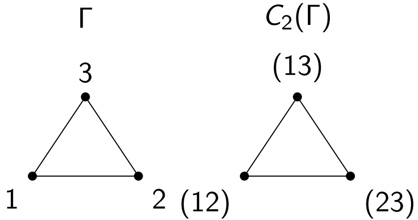
Exchange of 2 particles around loop c; one free phase ϕ_{c2}.



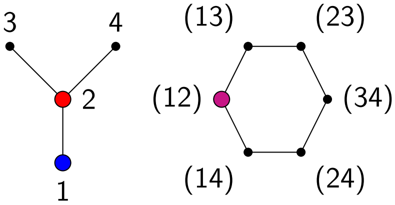
Exchange of 2 particles at Y-junction; one free phase ϕ_Y.

Basic cases

For 2 particles.

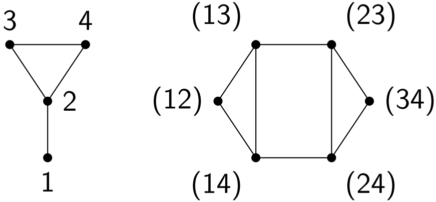


Exchange of 2 particles around loop c; one free phase ϕ_{c2}.



Exchange of 2 particles at Y-junction; one free phase ϕ_Y.

Lasso graph



Identify three 2-particle cycles:

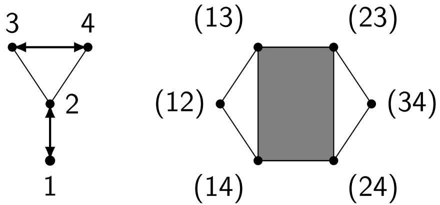
(i) Rotate both particles around loop c; phase $\phi_{c,2}$.

(ii) Exchange particles on Y-subgraph; phase ϕ_Y.

(iii) Rotate one particle around loop c other particle at vertex 1; $(1,2) \to (1,3) \to (1,4) \to (1,2)$, phase $\phi^1_{c,1}$.

Relation from contactable 2-cell $\phi_{c,2} = \phi^1_{c,1} + \phi_Y$.

Lasso graph



Identify three 2-particle cycles:

(i) Rotate both particles around loop c; phase $\phi_{c,2}$.

(ii) Exchange particles on Y-subgraph; phase ϕ_Y.

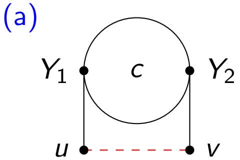
(iii) Rotate one particle around loop c other particle at vertex 1; $(1,2) \to (1,3) \to (1,4) \to (1,2)$, phase $\phi^1_{c,1}$.

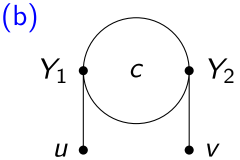
Relation from contactable 2-cell $\phi_{c,2} = \phi^1_{c,1} + \phi_Y$.

Let c be a loop. What is the relation between $\phi^u_{c,1}$ and $\phi^v_{c,1}$?

(a) u and v joined by path disjoint with c.
$\phi^u_{c,1} = \phi^v_{c,1}$ as exchange cycles homotopy equivalent.

(b) u and v *only* joined by paths through c.
Two lasso graphs so $\phi_{c,2} = \phi^u_{c,1} + \phi_{Y_1}$ & $\phi_{c,2} = \phi^v_{c,1} + \phi_{Y_2}$.
Hence $\phi^u_{c,1} - \phi^v_{c,1} = \phi_{Y_2} - \phi_{Y_1}$.

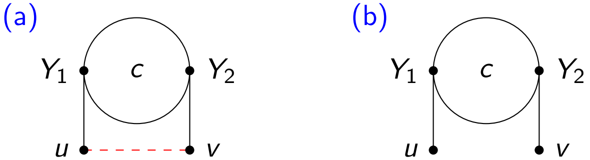




Let c be a loop. What is the relation between $\phi^u_{c,1}$ and $\phi^v_{c,1}$?

(a) u and v joined by path disjoint with c.
$\phi^u_{c,1} = \phi^v_{c,1}$ as exchange cycles homotopy equivalent.

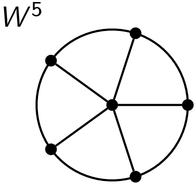
(b) u and v *only* joined by paths through c.
Two lasso graphs so $\phi_{c,2} = \phi^u_{c,1} + \phi_{Y_1}$ & $\phi_{c,2} = \phi^v_{c,1} + \phi_{Y_2}$.
Hence $\phi^u_{c,1} - \phi^v_{c,1} = \phi_{Y_2} - \phi_{Y_1}$.



- Relations between phases involving c encoded in phases ϕ_Y.
$H_1(C_2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$, where A determined by Y-cycles.
- In (a) we have a $\mathcal{B}$ subgraph & using (b) also $\phi_{Y_1} = \phi_{Y_2}$.

3-connected graphs

The prototypical 3-connected graph is a *wheel* W^k.



Theorem (Wheel theorem)

Let Γ be a simple 3-connected graph different from a wheel. Then for some edge $e \in \Gamma$ either $\Gamma \setminus e$ or Γ / e is simple and 3-connected.

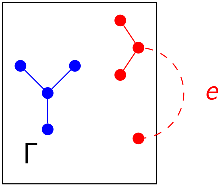
- $\Gamma \setminus e$ is Γ with the edge e removed.
- Γ / e is Γ with e contracted to identify its vertices.

Lemma

For 3-connected simple graphs all phases ϕ_Y are equal up to a sign.

Sketch proof. The lemma holds on K_4 (minimal wheel). By wheel theorem we need to show that adding an edge or expanding a vertex any new phases ϕ_Y are the same as an original phase.

Adding an edge: $\Gamma \cup e$

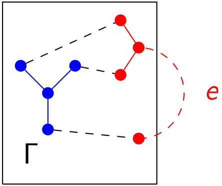


Lemma

For 3-connected simple graphs all phases ϕ_Y are equal up to a sign.

Sketch proof. The lemma holds on K_4 (minimal wheel). By wheel theorem we need to show that adding an edge or expanding a vertex any new phases ϕ_Y are the same as an original phase.

Adding an edge: $\Gamma \cup e$

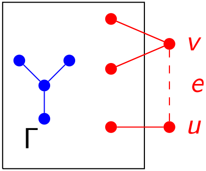


Using 3-connectedness identify independent paths in Γ to make $\mathcal{B}$. Then $\phi_Y = \phi_Y$.

Lemma

For 3-connected simple graphs all phases ϕ_Y are equal up to a sign.

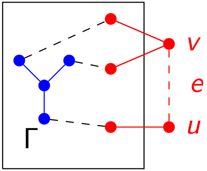
Sketch proof. The lemma holds on K_4 (minimal wheel). By wheel theorem we need to show that adding an edge or expanding a vertex any new phases ϕ_Y are the same as an original phase.
Vertex expansion: Split vertex of degree > 3 into two vertices u and v joined by a new edge e.



Lemma

For 3-connected simple graphs all phases ϕ_Y are equal up to a sign.

Sketch proof. The lemma holds on K_4 (minimal wheel). By wheel theorem we need to show that adding an edge or expanding a vertex any new phases ϕ_Y are the same as an original phase.
Vertex expansion: Split vertex of degree > 3 into two vertices u and v joined by a new edge e.



Using 3-connectedness identify independent paths in Γ to make $\mathcal{B}$.
Then $\phi_Y = \phi_Y$.

Theorem

For a 3-connected simple graph, $H_1(C_2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$*, where* $A = \mathbb{Z}_2$ *for non-planar graphs and* $A = \mathbb{Z}$ *for planar graphs.*

Theorem

For a 3-connected simple graph, $H_1(C_2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$*, where* $A = \mathbb{Z}_2$ *for non-planar graphs and* $A = \mathbb{Z}$ *for planar graphs.*

Proof.

- For K_5 and $K_{3,3}$ every phase $\phi_Y = 0$ or π. By Kuratowski's theorem a non-planar graph contains a subgraph which is isomorphic to K_5 or $K_{3,3}$.

Theorem

For a 3-connected simple graph, $H_1(C_2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$, where $A = \mathbb{Z}_2$ for non-planar graphs and $A = \mathbb{Z}$ for planar graphs.

Proof.

- For K_5 and $K_{3,3}$ every phase $\phi_Y = 0$ or π. By Kuratowski's theorem a non-planar graph contains a subgraph which is isomorphic to K_5 or $K_{3,3}$.
- For planar graphs the anyon phase can be introduced by drawing the graph in the plane and integrating the anyon vector potential $\frac{\alpha}{2\pi}\hat{z} \times \frac{r_1 - r_2}{|r_1 - r_2|^2}$ along the edges of the two-particle graph.

Classification of graph statistics

Ko & Park (2011)

$$H_1(C_n(G)) = \mathbb{Z}^{N_1(G)+N_2(G)+N_3(G)+\beta_1(G)} \oplus \mathbb{Z}_2^{N_3'(G)} \qquad (2)$$

- $N_1(G)$ sum over one cuts j of $N(n, G, j)$.

$$N(n,G,j) = \binom{n+\mu_j-2}{n-1}(\mu(j)-2) - \binom{n+\mu_j-2}{n} - (v_j - \mu_j - 1)$$

  μ_j # components of $G \setminus j$.
- $N_2(G)$ sum over two connected components of G.
- $N_3(G)$ # 3-connected planar components of G.
- $N_3'(G)$ # 3-connected non-planar components of G.
- $\beta_1(G)$ # of loops of G.

Summary

- Classification of abelian quantum statistics on graphs by graph theoretic argument.
- Physical insight into dependence of statistics on graph connectivity.
- Interesting new features of graph statistics.
- Are there phenomena associated with new forms of graph statistics - e.g. fractional quantum Hall experiment on network?

JH, JP Keating, JM Robbins and A Sawicki, "*n*-particle quantum statistics on graphs," *Commun. Math. Phys.* (2014) **330** 1293–1326 arXiv:1304.5781

JH, JP Keating and JM Robbins, "Quantum statistics on graphs," *Proc. R. Soc. A* (2010) doi:10.1098/rspa.2010.0254 arXiv:1101.1535