Existence of Ground State for the NLS on Star-like Graphs

A joint work in collaboration with D. Finco and D. Noja

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A Star-Like Graph

Metric graph: Each edge is associated either to a compact interval (if it is finite) or to $[0, +\infty)$ (if it is infinite)

$E$ : denotes the set of edges of $G$
$V$ : denotes the set of vertices of $G$

Assumption 1

$G$ is a connected graph with a finite number of edges and vertices, and it is composed by at least one infinite edge (one half-line) attached to a compact core.
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Notation

Hilbert space: \( \Psi \in L^2(G) \) means
\[
\Psi = (\psi_1, \psi_2, \ldots, \psi_{|E|}) \quad \psi_e \in L^2(I_e) \quad \forall e \in E
\]

Sobolev spaces:

\[ H^1(G) := \{ \Psi \in L^2(G) | \psi_e \in H^1(I_e) \forall e \in E \text{ and } \Psi \text{ is continuous in the vertices} \} \]
\[ H^2(G) := \{ \Psi \in H^1(G) | \psi_e \in H^2(I_e) \forall e \in E \} \]

Scalar products and norms are defined in a natural way:
\[
\| \Psi \|_G^2 = \sum_{e \in E} \| \psi_e \|_{I_e}^2
\]
The Nonlinear Schrödinger Equation

\[ i \frac{d}{dt} \Psi = H \Psi - |\Psi|^{2\mu} \Psi \quad 0 < \mu < 2 \]

**Linear term:** \( H \) is a linear operator with \( \delta \)-interaction in the vertices plus a potential

\[ \mathcal{D}(H) := \left\{ \Psi \in H^2(G) \mid \sum_{e \prec v} \partial_o \psi_e(v) = \alpha(v) \psi_e(v), \ \alpha(v) \in \mathbb{R}, \ \forall v \in V \right\}. \]

\[ H \Psi = -\Psi'' + W \Psi \]

**Nonlinear term:** Focusing powerlike nonlinearity, subcritical

**Componentwise:**

\[ i \frac{d}{dt} \psi_e = -\frac{d^2}{dx_e^2} \psi_e + W_e \psi_e - |\psi_e|^{2\mu} \psi_e \quad \forall e \in E \]

- \(|E|\) scalar equations
- Coupled by the conditions in the vertices
The ground state for the NLS

**Nonlinear energy functional:** Defined on $H^1(G)$ as

$$E[Ψ] = ||Ψ'||^2 + (Ψ, WΨ) + \sum_{v \in V} \alpha(v)|Ψ(v)|^2 - \frac{1}{\mu + 1} ||Ψ||^{2\mu+2}_{2\mu+2}$$

**Ground state:** Minimizer of $E[Ψ]$ at fixed mass $m = ||Ψ||^2$

**Problem:** Under what conditions on $G$, $H$, $m$ the ground state does/does not exist

or equivalently

Under what conditions on $G$, $H$, $m$ the infimum

$$\inf\{E[Ψ] | Ψ \in H^1(G), ||Ψ||^2 = m\}$$

is/is not attained
Orbital Stability of Ground State

Let $\hat{\Psi}$ be a ground state, and consider the Cauchy problem:

$$
\begin{cases}
  i \frac{d}{dt} \Psi = H \Psi - |\Psi|^{2\mu} \Psi \\
  \Psi|_{t=0} = \Psi_0
\end{cases}
$$

(1)

$e^{i\omega t} \hat{\Psi}$, $\omega \in \mathbb{R}$, is the stationary solution of (1) with initial datum $\Psi_0 = \hat{\Psi}$. 

\[\text{Diagram of orbits} \]
Orbital Stability of Ground State

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(1)

\( e^{i\omega t} \hat{\Psi} \), \( \omega \in \mathbb{R} \), is the stationary solution of (1) with initial datum \( \Psi_0 = \hat{\Psi} \).

Theorem (Cazenave Lions ’82)

Let \( 0 < \mu < 2 \). For any \( \varepsilon > 0 \) there exists \( \delta(\varepsilon) > 0 \) such that if

\[
\| \Psi_0 - \hat{\Psi} \|_{H^1} \leq \delta(\varepsilon)
\]

then the corresponding solution of (1) is such that

\[
\sup_{t \in \mathbb{R}^+} \inf_{\theta \in \mathbb{R}} \| \Psi(t) - e^{i\theta} \hat{\Psi} \|_{H^1} \leq \varepsilon
\]
Main Results: \( W = 0, \alpha = 0 \) [Adami-Serra-Tilli ’14 ’16]

- Take \( W = 0 \) and \( \alpha(v) = 0 \) for all \( v \in V \)
- Find topological and metric conditions on \( G \) that guarantee existence/non-existence of the ground state

**Condition H:** From every point of the graph one can get to infinity through two disjoint paths
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- Take $W = 0$ and $\alpha(v) = 0$ for all $v \in V$
- Find topological and metric conditions on $\mathcal{G}$ that guarantee existence/nonexistence of the ground state

**Condition H:** From every point of the graph one can get to infinity through two disjoint paths

**Adami-Serra-Tilli '14:** If (H) is satisfied the ground state does not exist unless $\mathcal{G}$ is isometric to a bubble tower.
Main Results: Star-Graph, $W = 0$, $\alpha < 0$ [Adami, C.C., Finco, Noja ’14]

- Consider a Star-Graph
- Take $W = 0$ and $\alpha(v) < 0$

Adami, C.C., Finco, Noja ’14: There exists $m^* > 0$ such that for $0 < m < m^*$ the ground state exists.

Adami, Noja, Visciglia ’13; Fukuizumi, Ohta, Ozawa ’08: If $|E| = 2$ the ground state exists for any $m > 0$.

Adami, C.C., Finco, Noja ’16: If $|E| \geq 3$ there exists $m^{**} > 0$ such that for $m > m^{**}$ the ground state does not exist.
Assumption 2

\[ W = W_+ - W_- \text{ with } W_\pm \geq 0, \; W_+ \in L^1(G) + L^{\infty}(G), \; \text{and } W_- \in L^r(G) \text{ for some } r \in [1, 1 + 1/\mu]. \]

Assumption 3

\[ \inf \sigma(H) := -E_0, \; E_0 > 0 \text{ and it is an isolated eigenvalue.} \]

Theorem

Let \( 0 < \mu < 2 \). If Assumptions 1, 2, and 3 hold true then

\[ -\infty < \inf \{E[\Psi] | \Psi \in H^1(G), \|\Psi\|^2 = m\} \leq -E_0 m \]

for any \( m > 0 \). Moreover, there exists \( m^* > 0 \) such that for \( 0 < m < m^* \) the infimum is attained, i.e., the ground state exists.
Main Results: Generic Starlike-Graph [C.C., Finco, Noja prep. ’16]

The inequality
\[
\inf \left\{ E[\Psi] \mid \Psi \in H^1(G), \|\Psi\|^2 = m \right\} \leq -E_0 m
\]
is a direct consequence of
\[
E[\Psi] \leq \|\Psi\|^2 + (\Psi, W\Psi) + \sum_{v \in V} \alpha(v)|\Psi(v)|^2
\]
for all \( \Psi \in H^1(G) \) and of
\[
-E_0 m = \inf \left\{ \|\Psi\|^2 + (\Psi, W\Psi) + \sum_{v \in V} \alpha(v)|\Psi(v)|^2 \mid \Psi \in H^1(G), \|\Psi\|^2 = m \right\}
\]
Concentration-Compactness

For any $\Psi \in L^2(G)$ define the concentration function
\[ \rho(\Psi, s) = \sup_{y \in G} \| \Psi \|^2_{L^2(B_G(y,s))}. \]

Let $\{\Psi_n\}_{n \in \mathbb{N}}$ be such that: $\Psi_n \in H^1(G)$,
\[ \|\Psi_n\|^2 = m \quad \sup_{n \in \mathbb{N}} \|\Psi_n'\| < C \]

Define the concentrated mass parameter $\tau$ as
\[ \tau = \lim_{s \to \infty} \lim_{n \to \infty} \rho(\Psi_n, s). \]

i) (Compactness) If $\tau = m$, at least one of the two following cases occurs:
   i$_1$) (Convergence) There exists a function $\Psi \in H^1(G)$ such that $\Psi_n \to \Psi$ in $L^p(G)$ for all $2 \leq p \leq \infty$.
   i$_2$) (Runaway) $\|\Psi_n\|_{L^p(B_G(y,s))} \to 0$ for all $2 \leq p \leq \infty$, $y \in G$, $s > 0$.

ii) (Vanishing) If $\tau = 0$, then $\Psi_n \to 0$ in $L^p(G)$ for all $2 < p \leq \infty$.

iii) (Dichotomy) If $0 < \tau < m$, then there exist two sequences $\{R_n\}_{n \in \mathbb{N}}$ and $\{S_n\}_{n \in \mathbb{N}}$ in $H^1(G)$ such that: $\|\Psi_n - R_n - S_n\| \to 0$,
\[ \|R_n\|^2 \to \tau \quad \|S_n\|^2 \to m - \tau \]
\[ \text{dist}(\text{Supp} \ R_n, \text{Supp} \ S_n) \to \infty \]
Concentration-Compactness

If $\Psi_n$ is a minimizing sequence

- Vanishing and Dichotomy cannot occur

- If $i_2$ (Runaway), then

$$\lim_{n \to \infty} E[\Psi_n] \geq -\gamma \mu m^{1+\frac{2\mu}{2-\mu}}.$$ 

- $-\gamma \mu m^{1+\frac{2\mu}{2-\mu}}$ is the energy of the ground state of mass $m$ of the NLS on the real line

$$-\gamma \mu m^{1+\frac{2\mu}{2-\mu}} = \inf_{\psi \in H^1(\mathbb{R}), \|\psi\|_{L^2(\mathbb{R})} = m} \left( \|\psi\|_{L^2(\mathbb{R})}^2 - \frac{1}{\mu + 1} \|\psi\|_{L^{2\mu+2}(\mathbb{R})}^{2\mu+2} \right)$$
Concentration-Compactness

If $\Psi_n$ is a minimizing sequence

- Vanishing and Dichotomy cannot occur
- If $i_2)$ (Runaway), then

$$\lim_{n \to \infty} E[\Psi_n] \geq -\gamma \mu m^{1+\frac{2\mu}{2-\mu}}.$$ 

- But for $m$ small enough

$$\inf \left\{ E[\Psi] \mid \Psi \in H^1(\mathcal{G}), \|\Psi\|^2 = m \right\} \leq -E_0 m < -\gamma \mu m^{1+\frac{2\mu}{2-\mu}}$$

- Indeed $m^* = \left( \frac{E_0}{\gamma \mu} \right)^{\frac{1}{\mu}} - \frac{1}{2}$
Bifurcation Analysis

If $E_0$ is a simple eigenvalue one can use bifurcation theory to find a candidate. Consider the stationary equation

$$H\Phi - |\Phi|^{2\mu}\Phi = -\omega\Phi \quad \Phi \in D(H), \, \omega \in \mathbb{R}$$

Let $H\Phi_0 = -E_0\Phi_0$, with $\|\Phi_0\|^2 = 1$. Then for $\omega > E_0$ there exists a solution

$$\Phi(\omega) = a_*(\omega)\Phi_0 + \Theta_*(a_*(\omega), \omega)$$

such that

$$m(\omega) = \|\Phi(\omega)\|^2 = \left(\frac{\omega - E_0}{\|\Phi_0\|^{2\mu+2}}\right)^{\frac{1}{\mu}} + o\left((\omega - E_0)^{\frac{1}{\mu}}\right)$$

$$\|\Phi(\omega)\|^2$$

$\uparrow$

$E_0 \quad \omega$

$$E[\Phi(\omega(m))] = -E_0m + o(m)$$
Remarks

- A sufficient condition to have an isolated eigenvalue is
  \[ \int_G W \, dx + \sum_{v \in V} \alpha(v) < 0 \]

- If \( W = 0 \) and \( \alpha(v) \leq 0 \ \forall v \in V \), and strictly negative for at least one vertex. Then \(-E_0\) is a simple eigenvalue [Exner, Jex ’12].

- For compact graphs with \( \delta \)-vertices, simplicity of the spectrum can be achieved by small modification of edge lengths [Berkolaiko, Liu ’16].

- The analysis can be extended in principle to the case in which \(-E_0\) has multiplicity larger than one. One has to use bifurcation analysis in the degenerate case.

- We do not claim that \( \Phi(\omega(m)) \) is the ground state, even though we conjecture that this is true. This can be proved in the case of the Star-Graph with \( W = 0 \) and \( \alpha(v) < 0 \).

- Our result does not cover the case \( W = 0, \alpha(v) = 0 \), treated by Adami, Serra, Tilli. Since in this case there are not isolated eigenvalues, \( \sigma(H) = \sigma_{ess}(H) = [0, +\infty) \).
Global Well-Posedness in $H^1(G)$

**Theorem (Global Well-Posedness)**

Let $0 < \mu < 2$. For any $\Psi_0 \in H^1(G)$, the Cauchy problem

$$
\begin{cases}
    i \frac{d}{dt} \Psi = H \Psi - |\Psi|^{2\mu} \Psi \\
    \Psi|_{t=0} = \Psi_0
\end{cases}
$$

has a unique weak solution $\Psi \in C^0([0, \infty), H^1(G)) \cap C^1([0, \infty), H^1(G)^*)$.

The proof uses the following conservation laws

**Proposition (Conservation laws)**

Let $\mu > 0$. For any weak solution $\Psi \in C^0([0, T), H^1(G)) \cap C^1([0, T), H^1(G)^*)$ to the problem (2), the following conservation laws hold at any time $t$:

$$
\|\Psi(t)\|^2 = \|\Psi_0\|^2, \quad E[\Psi(t)] = E[\Psi_0].
$$

Together with Local Well-posedness (proved by Banach fixed point theorem) and Gagliardo-Nirenberg inequalities.
Proposition

Let $\mathcal{G}$ be a graph with a finite number of edges and vertices. Then if $p, q \in [2, +\infty]$, with $p \geq q$, and $\alpha = \frac{2}{2+q}(1 - q/p)$, there exists $C$ such that

$$\|\Psi\|_p \leq C\|\Psi\|_{H^1}^\alpha \|\Psi\|_q^{1-\alpha},$$

for all $\Psi \in H^1(\mathcal{G})$.

If the $\mathcal{G}$ has one or more infinite edges one has the stronger inequality

$$\|\Psi\|_p \leq C\|\Psi'\|^{\alpha} \|\Psi\|_q^{1-\alpha},$$

See, e.g., [Mugnolo Springer ’14, Adami-Serra-Tilli JFA ’16].