Spectras of interacting particles on quantum graphs

QMath 13

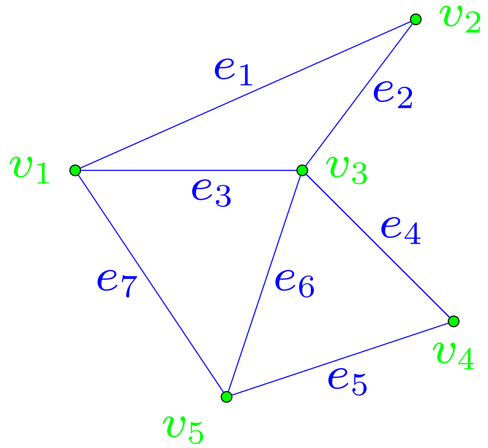
Atlanta, 8-11 October 2016

Jens Bolte
Royal Holloway, University of London

Based on joint work with J. Kerner, G. Garforth

Quantum graphs

Quantum graph: *one* particle moving along edges of a finite, metric graph.



A graph with $V = 5$ vertices and $E = 7$ edges

The (simplest) one-particle Hamiltonian is a Laplacian, describing a free particle on the graph. It acts on the edge-e component as

$$(-\Delta_1 \psi)_e(x_e) = -\frac{\partial^2 \psi_e}{\partial x_e^2}(x_e) \ ,$$

plus (self-adjoint) boundary conditions in the vertices.

There is a close analogy to Laplacians on manifolds. Many *spectral properties* are known, including

- Eigenvalue count follows a Weyl law
- Eigenvalue correlations (empirically) follow RMT predictions
- Trace formulae
- Periodic orbit correlations (heuristically) leading to eigenvalues correlations
- Inverse problems, isospectrality
- Nodal domains of eigenfunctions

Many-particle quantum systems on graphs are, in comparison, still less studied.

Our goals:

- Construct models with two-particle interactions
- Study basic spectral properties: discreteness of spectra, Weyl law etc.
- Identical particles: bosons, fermions
- (Bose-Einstein condensation)
- Secular equation, eigenvalues, spectral statistics

Other topics include:

- Many-particle statistics (Harrison, Keating, Robbins, Sawicki 2010-13)
- Non-linear (Schrödinger or Gross-Pitaevskii) equations (Adami et al 2010-13)
- Anderson localisation (Sabri 2012-13)
- Trace formulae
- Other types of operators: Schrödinger, Pauli, Dirac

Basic constructions

One-particle Hilbert space

$$\mathcal{H}_1 = L^2(\Gamma) = \bigoplus_{e=1}^{E} L^2(0,l_e) = \left\{\psi = (\psi_1,\dots,\psi_E);\psi_e \in L^2(0,l_e)\right\} ,$$

and similarly Sobolev spaces $H^m(\Gamma)$. Boundary values (in edge ends),

$$\begin{aligned} \psi_{bv} &= \big(\psi_1(0),\dots,\psi_E(0),\psi_1(l_1),\dots,\psi_E(l_E)\big)^T \\ \psi'_{bv} &= \big(\psi'_1(0),\dots,\psi'_E(0),-\psi'_1(l_1),\dots,-\psi'_E(l_E)\big)^T . \end{aligned}$$

Linear maps on the space $\mathbb{C}^{2E}$ of boundary values:

- Projector P_1,
- L_1 self-adjoint on $\mathrm{ran}\, P_1^{\perp}$.

Self-adjoint realisations of the Laplacian are in one-to-one correspondence to closed, semi-bounded quadratic forms.

Theorem [Kuchment 04]
The quadratic form

$$Q^{(1)}_{P_1,L_1}[\psi] = \sum_{e=1}^{E} \int_0^{l_e} |\psi_e'(x)|^2 \, dx - \big(\psi_{bv}, L_1\psi_{bv}\big)_{\mathbb{C}^{2E}} ,$$

with domain

$$\mathcal{D}_{Q^{(1)}} = \{\psi \in H^1(\Gamma);\; P_1\psi_{bv} = 0\} ,$$

is associated with the one-particle Laplacian on the domain

$$\mathcal{D}_1(P_1, L_1) = \{\psi \in H^2(\Gamma);\; P_1\psi_{bv} = 0 \text{ and } P_1^{\perp}\psi_{bv}' + L_1 P_1^{\perp}\psi_{bv} = 0\} .$$

Scattering approach

Eigenvalues k^2 of the Laplacian as zeros of a secular determinant:

$$\det\bigl(\mathbb{1} - U(k)\bigr) = 0,$$

where $U(k) = S(k)T(k)$ is a unitary $2E \times 2E$ matrix with

$$S(k) = -\bigl(P + L + ikP^{\perp}\bigr)^{-1}\bigl(P + L + ikP^{\perp}\bigr) \quad \text{and} \quad T(k) = \begin{pmatrix} 0 & e^{ik\boldsymbol{l}} \\ e^{ik\boldsymbol{l}} & 0 \end{pmatrix}$$

- Laplacian with Neumann b.c.: Kottos, Smilansky (1997)
- Laplacian with general self adjoint b.c.: Kostrykin, Schrader (2006)
- Schrödinger operators $-\Delta + V$: Rueckriemen, Smilansky (2012), JB, Egger, Rueckriemen (2015)

Many-particle systems

From $\mathcal{H}_1$ one constructs the N-particle Hilbert space

$$\mathcal{H}_N = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1 \ ,$$

so that N-particle states are functions $\Psi = (\psi_{e_1 \dots e_N})$ with

$$\psi_{e_1 \dots e_N} \in L^2\big(D_{e_1 \dots e_N}\big) \ , \quad \text{where } D_{e_1 \dots e_N} = [0, l_{e_1}] \times \cdots \times [0, l_{e_N}] \ .$$

In the following we shall restrict ourselves to $N = 2$ and $E = 1$, i.e., two particles on an interval. The configuration space then is $D = (0, l) \times (0, l)$.

We construct rigorous versions of δ-type contact interactions,

$$H = -\Delta_2 + \alpha(x)\, \delta(x - y) \ .$$

The contact interactions require jump conditions on the normal derivative across the diagonal $x = y$. In more detail:

Dissect the square D as $D^* = D_- \cup D_+$ along $x = y$ and introduce functions

$$\psi^\pm : \; D_\pm \to \mathbb{C} \; , \quad \text{such that} \quad \psi(x,y) = \begin{cases} \psi^+(x,y) \; , & x > y \\ \psi^-(x,y) \; , & x < y \end{cases} \; .$$

Use boundary values

$$\psi_{bv}(y) = \begin{pmatrix} \psi^-(0,y) \\ \psi^+(l,y) \\ \psi^+(y,0) \\ \psi^-(y,l) \\ \psi^+(y,y) \\ \psi^-(y,y) \end{pmatrix} \quad \text{and} \quad \psi'_{bv}(y) = \begin{pmatrix} \psi^-_x(0,y) \\ -\psi^+_x(l,y) \\ \psi^+_y(y,0) \\ -\psi^-_y(y,l) \\ (\psi^+_x - \psi^+_y)(y,y) \\ (\psi^-_x - \psi^-_y)(y,y) \end{pmatrix}$$

along the six sides of the triangles D_- and D_+.

Split the space of boundary values as

$$\mathbb{C}^6 = V_{vertex} \oplus V_{contact} \ .$$

Choose 'non-interacting' P_{vertex} and L_{vertex}, as well as

$$P_{contact}(y) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \ , \qquad L_{contact}(y) = -\frac{1}{2}\alpha(y)\mathbb{1}_2 \ ,$$

and set up a quadratic form,

$$Q^{(2)}_{P,L}[\psi] = \langle \nabla\psi, \nabla\psi \rangle_{L^2(D^*)} - \int_0^l \langle \psi_{bv}(y), L(y)\psi_{bv}(y) \rangle_{\mathbb{C}^6} \, \mathrm{d}y \ ,$$

with domain

$$\mathcal{D}_{Q^{(2)}} = \{\psi \in H^1(D^*);\ P(y)\psi_{bv}(y) = 0 \ \forall y \in [0,l]\} \ .$$

This means that $\psi(x,y)$ is continuous across $x=y$, but the normal derivative jumps by α,

$$\psi_x^+ - \psi_y^+ - \alpha\psi^+ = \psi_x^- - \psi_y^- .$$

Theorem [JB, Kerner 2013]
The quadratic form $Q^{(2)}_{P,L}$ on the domain $\mathcal{D}_{Q^{(2)}}$ is closed and semi-bounded. The associated self-adjoint operator is the Laplacian $-\Delta_2$ on the domain

$$\mathcal{D}_2(P,L) := \{\psi \in H^2(D);\ P(y)\psi_{bv}(y) = 0 \text{ and } P^{\perp}(y)\psi'_{bv}(y) + L(y)P^{\perp}(y)\psi_{bv}(y) = 0\} .$$

A basic spectral property of the operators is the following.

Theorem [JB, Kerner 2013]
The operator $(-\Delta_2, \mathcal{D}_2(P, L))$ *has a compact resolvent. In particular, it possesses a discrete spectrum and the eigenvalue count follows a Weyl law,*

$$\#\{n \in \mathbb{N};\ \lambda_n \leq \lambda\} \sim \frac{\mathcal{L}^2}{4\pi}\lambda\ , \quad \lambda \to \infty\ .$$

Identical particles

For two particles the bosonic/fermionic projector is

$$(\Pi_{B/F}\psi)(x,y) = \frac{1}{2}\big(\psi(x,y) \pm \psi(y,x)\big) \ .$$

- It can be arranged that $[H_2, \Pi_B] = 0$, hence the operator has a bosonic version $H_{2,B}$.
- There is an immediate generalisation to arbitrary compact graphs.
- The operator can be promoted to an operator for N bosons,

$$H_{N,B} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} \alpha(x_i)\delta(x_i - x_j) \ .$$

This yields an extension of the **Lieb-Liniger model** from a circle to a metric graph.

Hardcore-limit $\alpha \to \infty$: Dirichlet conditions at $x_i = x_j$ (**Tonks-Girardeau gas**).

Solvable models

(This is recent work with George Garforth: arXiv:1609.00828v1)

For one particle, eigenfunctions are of the form

$$a\,e^{ikx} + b\,e^{-ikx}$$

on each edge. This is used to produce the finite-dimensional secular determinant. For two-particles, the general form is

$$\int_{\mathbb{R}^2} a(k_1, k_2)\, e^{i(k_1 x_1 + k_2 x_2)}\; dk_1\; dk_2\ .$$

However, in some cases a **Bethe ansatz** for the eigenfunctions,

$$\sum_{k_1, k_2} A(k_1, k_2)\, e^{i(k_1 x_1 + k_2 x_2)}$$

(finite sum) will be possible. Such cases are called solvable.

Early examples are:

- δ-interacting particles on a circle: Lieb, Liniger (1963)
- δ-interacting particles on an interval: Yang (1967), Gaudin (1971)
- $\tilde{\delta}$-interacting particles on a star graph with two infinite edges Caudrelier, Crampe (2007)

The $\tilde{\delta}$-interaction on the star graph with two infinite edges is formally given as

$$\alpha\big(\delta(x_1 - x_2) + \delta(x_1 + x_2)\big).$$

It acts whenever $x_1 = \pm x_2$, i.e., when both particles are located the same distance away from the vertex, either on the same or on different edges.

A similar interaction can be defined on any graph and can be made rigorous in close analogy to the δ-interactions. When e, e' are two edges emanating from the same vertex, then

$$\psi^{+}_{ee'} = \psi^{-}_{e'e}$$

across $x = y$, and

$$\partial_x \psi^+_{ee'} - \partial_y \psi^+_{ee'} - \alpha \psi^+_{ee'} = \partial_x \psi^-_{e'e} - \partial_y \psi^-_{e'e}.$$

The Bethe ansatz then is of the form

$$\sum_{\sigma \in \mathcal{W}_2} A_\sigma \, e^{i(k_{\sigma(1)} x_1 + k_{\sigma(2)} x_2)},$$

where $\mathcal{W}_2$ is a finite group (of order 8) generated by I, T, R with relations

1. $TT = I$,
2. $RR = I$,
3. $TRTR = RTRT$.

We applied this in two examples:

- Equilateral star graph with DFT central scattering matrix
- Tetrahedron with rationally independent edge lengths

In both types of examples we found a secular equation,

$$\det\left(\mathbb{1} - U(k_1, k_2)\right) = 0,$$

for the eigenvalues $k_1^2 + k^2$, where

$$U(k_1, k_2) = E(k_2)Y(k_2 - k_1)(\mathbb{1}_2 \otimes S(k_2) \otimes \mathbb{1}_{2E})Y(k_1 + k_2),$$

and

$$Y(k) = \frac{1}{k + i\alpha}\begin{pmatrix} -i\alpha & k \\ k & -i\alpha \end{pmatrix} \otimes \boldsymbol{\alpha} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes (\mathbb{1}_{E^2} - \boldsymbol{\alpha})\mathbb{T}_{E^2}$$

$$E(k) = \mathbb{1}_{4E} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \boldsymbol{e^{ikl}};$$

$\mathbb{T}_{E^2}$ is a permutation matrix.

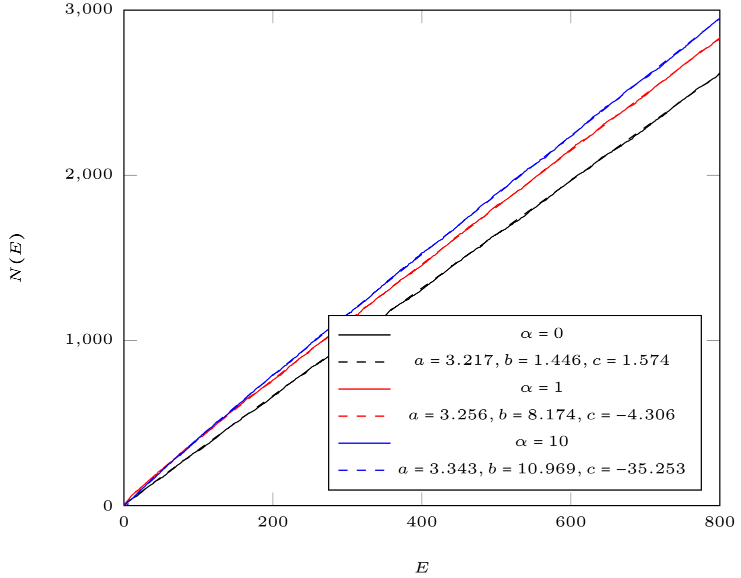


Abbildung 1: Eigenvalue counting function $N(E)$ for two bosons on a 9-edge equilateral star

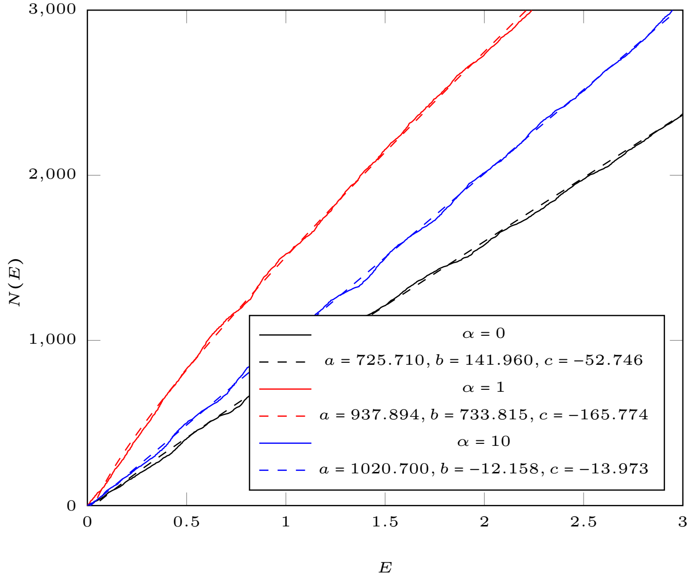


Abbildung 2: Eigenvalue counting function $N(E)$ for two bosons on a tetrahedron

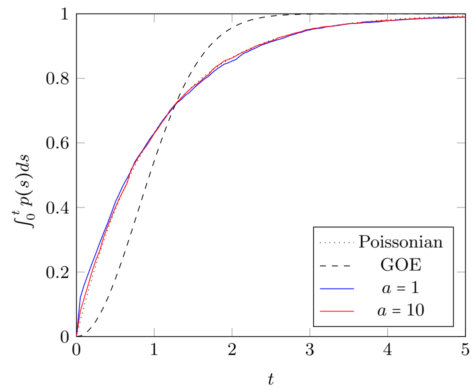


Abbildung 3: Integrated level spacings distributions for systems of two bosons on a tetrahedron (first 3000 eigenvalues).